Determinants

1. **Permutations**. Suppose that $(j_1, j_2, \ldots, j_n)$ is a *permutation* of $(1, 2, \ldots, n)$, that is each j_k is one of the integers between 1 and n, and every such integer occurs exactly once. Every permutation can be obtained starting from $(1, 2, \ldots, n)$ by consequtive interchanges of pairs of numbers. Such interchange of two numbers is called a *transposition*. For example, to obtain $(5, 3, 2, 4, 1)$ we start with $(1, 2, 3, 4, 5)$, and then

- interchange 1 and 5 to obtain $(5, 2, 3, 4, 1)$, and then
- interchange 2 and 3 to obtain $(5, 3, 2, 4, 1)$.

The number of required transpositions to obtain a given permutation may depend on the way we do it, but the *parity* of this number depends only on this given permutation. Thus a permutation is called *even* if an even number of transpositions is required, and *odd* otherwise. For example,

- the identity permutation $(1, 2, \ldots, n)$ is even (it is obtained using 0 transpositions),
- every transposition itself is odd,
- $(5, 3, 2, 4, 1)$ is even (because we obtained it above with two transpositions).

Here is the complete list of even permutations on 3 elements: $(1, 2, 3), (2, 3, 1), (3, 1, 2)$. The rest are odd.

The total number of permutations of n elements is $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$, exactly half of them are even and the rest are odd.

2. **Definition of a determinant**. To each *square* matrix A corresponds a number, called *determinant* of A, and denoted by $|A|$ or det A. For an $n \times n$ matrix $A = (a_{ij})$ we define

$$\det A = \sum_{(j_1, \ldots, j_n)} \pm a_{1j_1} a_{2j_2} \ldots a_{nj_n},$$

where the summation is over all permutations $(j_1, \ldots, j_n)$ of $(1, 2, \ldots, n)$, and the $+$ sign is chosen for each even permutation, and the $-$ sign for each odd permutation. For example,

if $n = 1$, then $\det A = a_{11}$,
if $n = 2$, then $\det A = a_{11}a_{22} - a_{12}a_{21}$,
if $n = 3$, then $\det A =$

$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}.$$

Thus there are $n!$ summands in the definitiuon of determinant of a $n \times n$ matrix. Fortunately there are simpler ways to evaluate determinants then applying the definition directly.

3. **Why do we need them?** There are two main uses of determinants.

a) A matrix A is singular if and only if $\det A = 0$. Thus we have an analytic criterion for this important property of a matrix.

b) The volume of the parallelepiped generated by n vectors $\mathbf{x_1}, \ldots, \mathbf{x_n}$ in $\mathbf{R^n}$ is equal to $|\det A|$, where $A = [\mathbf{x_1}, \ldots, \mathbf{x_n}]$ is the matrix, whose columns are these vectors. (It should be clear what does it mean a "parallelogram, generated by two vectors in $\mathbf{R}^2$", and a "parallelepiped, generated by three vectors in $\mathbf{R}^3$". The general definition, not appealing to geometric intuition and thus applicable in any dimension, is "the set of all linear combinations $\sum c_j \mathbf{x}_j$, with $0 \le c_j \le 1$.)

4. **Properties of determinants**.
(i) $\det AB = \det A \det B$.
(ii) $\det A = \det A^t$,
(iii) if two rows are interchanged, the determinant changes sign,
(iv) Determinant is a linear function with respect to each row. This means precisely the following. Suppose that we have 3 matrices $A = (a_{ij}), B = (b_{ij})$ and $C = (c_{ij})$, and a number k, such that for all $i \ne k$ we have $a_{ij} = b_{ij} = c_{ij},\ 1 \le j \le n$, and $a_{kj} = \alpha b_{kj} + \beta c_{kj},\ 1 \le j \le n$. Then

$$\det A = \alpha \det B + \beta \det C.$$

In particular, it follows from (iv) and (iii) that:
(v) If a row is multiplied by a number, the determinant is multiplied by the same number, and
(vi) if a multiple of a row is added to another row, the determinant does not change.

It follows from (ii) that operations on *columns* similar to (iii), (v), (vi) have similar effect to the row operations.

One can compute determinants using (iii), (v), (vi) or similar properties of column operations. For example

$$\begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 0 & -7 \end{vmatrix} = (-7) \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} = (-7) \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = -7.$$

(If one arrives in the end to something else than the identity matrix, the determinant is zero.) One derives from (v) that if some row consists entirely of zeros, then the determinant is zero. From (iii) follows that if two rows are equal, then determinant is zero. Moreover, if two rows are proportional, then determinant is zero. Using (ii) one obtains similar properties of columns.

5. **Row and column expansions**. Suppose that A is a $n \times n$ matrix. If we remove some $n - m$ rows and $n - m$ columns, where $m < n$, what remains is a new matrix of smaller size $m \times m$. Determinants of such matrices are called *minors* of order m of A.

The following special case is important. Take an element a_{ij} and remove the i-th row and j-th column from A. The minor we obtain is called the *minor of* a_{ij} and denoted by M_{ij}. (It is the determinant of some matrix of size $(n-1) \times (n-1)$.) Now the *cofactor of* a_{ij} is defined as $(-1)^{i+j} M_{ij}$. The following row expansion and column expansion formulas hold:

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij} \quad \text{for every} \quad i = 1, \ldots, n,$$

and

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij} \quad \text{for every} \quad j = 1, \ldots, n.$$

Your book has plenty of examples illustrating these formulas. They are convenient to use for evaluating determinants of matrices which have many zeros in some row or in some column. For a generic matrix the method based on row operations is faster.

6. **More applications of determinants**. Determinants permit to write simple explicit formulas for solutions of the problems we considered before in this course. We already mentioned the criterion of singularity $\det A = 0$. Here is more:

a) The inverse B of a non-singular matrix A is given by

$$b_{ij} = \frac{1}{|A|}(-1)^{i+j} M_{ji}.$$

Notice that the indices of the minor are in the opposite order! In words: to compute the inverse, we first replace each element of a matrix A by its cofactor, then transpose the result, and then divide it by $\det A$. We can divide by $\det A$ exactly because A is non-singular.

b) Solution of a system of linear equations

$$A\mathbf{x} = \mathbf{b}$$

with a non-singular matrix A. We know that in this case a solution exists and it is unique. For $j = 1, \ldots, n$, let A_j be the matrix obtained by replacing the j-th column of A with the column $\mathbf{b}$, that is with the RHS of the equation. Then the coordinates of the solution vector $\mathbf{x}$ are given by

$$x_j = \frac{\det A_j}{\det A}.$$

Again, these formulas have sense, because we assume A to be non-singular, so $\det A \neq 0$.

Explicit formulas in a) and b) are rarely used in numerical computation because there are much better algorithms, for example, the row operations. Still we will see that in many cases it is important to have an explicit formula for the solution of a problem.