DIRECT SINGULARITIES AND COMPLETELY INVARIANT DOMAINS OF ENTIRE FUNCTIONS

WALTER BERGWEILER AND ALEXANDRE EREMENKO

Abstract. Let f be a transcendental entire function and $a \in \mathbb{C}$. Suppose that a is a critical value of f or that f^{-1} has a direct singularity over a. We show that if D is a simply connected domain which does not contain a, then the full preimage $f^{-1}(D)$ is disconnected. Conversely, we show that if a is not a critical value and if f^{-1} does not have a direct singularity over a, then there exists a simply connected domain D not containing a whose full preimage is connected.

1. Introduction and results

The question considered in this paper is motivated by dynamics of entire functions [3, 6]. A component D of the Fatou set of an entire function f is called a *completely invariant domain* if $f^{-1}(D) = D$. This is a stronger property than simple invariance $f(D) \subset D$.

In what follows, all entire functions are assumed to be transcendental. It follows from a result of Baker [2, Theorem 1] that all invariant components of the Fatou set of such a function are simply connected. Baker also proved that at most one completely invariant domain can exist [1], and if f has a completely invariant domain, then all critical values (and thus all critical points) of f are contained in it [2, Theorem 2].

In [6, Lemma 11], the latter result of Baker was extended to the logarithmic singularities of f^{-1}: a completely invariant domain must contain all logarithmic singularities of f^{-1}. The authors of [6] stated without proof that the same is true for a wider class of singularities, namely for *direct singularities* of f^{-1}. (Iversen's classification of singularities of f^{-1} will be explained shortly). Their proof for logarithmic singularities does not extend to this more general case. On the other hand, a singularity of f^{-1} can be on the boundary of a completely invariant domain. Such examples were constructed in [4].

Date: July 31, 2006.

1991 *Mathematics Subject Classification.* 30D20.

The first author was supported by the Alexander von Humboldt Foundation and by the G.I.F., the German–Israeli Foundation for Scientific Research and Development, Grant G-809-234.6/2003.

The second author was supported by NSF grants DMS-0555279 and DMS-0244547.

In this paper, we give a new characterization of direct singularities, from which it will follow that all direct singularities of f^{-1} lie in the completely invariant domain, if an entire function f has such a domain.

We first recall Iversen's classification of singularities; see [5], [7] or [8, p. 289]. Let f be a transcendental entire function and $a \in \mathbb{C}$. Consider the open discs $B(a,r)$ of radius r centered at a. For every $r > 0$, let us choose a component U_r of the preimage $f^{-1}(B(a,r))$ in such a way that $r_1 < r_2$ implies $U_{r_1} \subset U_{r_2}$. Then we have two possibilities:

a) $\bigcap_{r>0} U_r$ consists of one point, or

b) $\bigcap_{r>0} U_r = \emptyset$.

In the latter case we say that our choice $r \mapsto U_r$ defines a *transcendental singularity* of f^{-1} over a. We also say that a is the *projection* of the transcendental singularity, and any of the sets U_r is called a *neighborhood* of the transcendental singularity. A transcendental singularity over a is called *direct* if for some $r > 0$ we have $f(z) \neq a$ for $z \in U_r$. Otherwise it is called *indirect.* A direct singularity is called *logarithmic* if the restriction $f : U_r \to B(a,r)\backslash\{a\}$ is a universal covering for some $r > 0$.

For example, $\exp z$ has a logarithmic singularity over 0, and $(\sin z)/z$ has two indirect singularities over 0. The function

$$f(z) = \exp\left(\sum_{k=1}^{\infty}\left(\frac{z}{2^k}\right)^{2^k}\right) \tag{1}$$

has infinitely many direct singularities over 0, but none of them is logarithmic. This example and some further results concerning direct singularities of entire functions will be discussed in §3.

Suppose that an entire function f has no critical values and no direct singularities over some point a. This statement is equivalent to the following: if $r > 0$, then every component of the preimage $f^{-1}(B(a,r))$ contains a simple a-point of f.

Theorem. *Let f be a transcendental entire function and $a \in \mathbb{C}$.*

a) *If a is a critical value or f^{-1} has a direct singularity over a, then for every simply connected region $D \subset \mathbb{C}$ such that $a \notin D$, the full preimage $f^{-1}(D)$ is disconnected.*

b) *If a is not a critical value and f^{-1} has no direct singularities over a, then there exists a simply connected domain $D \subset \mathbb{C}$ such that $a \notin D$ and the full preimage $f^{-1}(D)$ is connected.*

Corollary. *Let f be a transcendental entire function and D a completely invariant domain of f. Then D contains all critical values of f and all projections of direct singularities of f^{-1}.*

2. Proof of the Theorem

We shall repeatedly use the following result of Iversen [7], which follows easily from the Gross star theorem [8, p. 292], or from the variant of the Gross star theorem stated as Lemma 2 below.

Lemma 1. *Let ϕ be a holomorphic branch of the inverse f^{-1} defined in a neighborhood of some point w_0 and let $\gamma : [0,1) \to \mathbb{C}$ be a curve with $\gamma(0) = w_0$. Then for every $\varepsilon > 0$ there exists a curve $\tilde{\gamma} : [0,1) \to \mathbb{C}$ satisfying $\tilde{\gamma}(0) = w_0$ and $|\gamma(t) - \tilde{\gamma}(t)| < \varepsilon$ such that ϕ has an analytic continuation along $\tilde{\gamma}$.*

Proof of part a) of the Theorem. We may assume without loss of generality that $a \in \partial D$, by enlarging D if necessary. Assume that the full preimage $f^{-1}(D)$ is connected.

Suppose first that a is a critical value. (Actually Baker's proof applies to this case, but we include a proof similar to our proof for the case of a direct singularity.) Let b be a point such that $f(b) = a$ and $f'(b) = 0$. Then there is a disc $B(a,r)$ and a component V of $f^{-1}(B(a,r))$ such that V contains b, and the restriction $f : \overline{V} \to \overline{B(a,r)}$ is a ramified covering of degree $m \geq 2$, ramified only at b. Choosing r small enough, we can assure that there is a point $w_0 \in \partial B(a,r) \cap D$. Choose a preimage $z_0 \in \partial V$ of the point w_0. Then some open arc $\gamma \subset \partial V$ with endpoints z_0 and z_1 is mapped homeomorphically onto $f(\gamma) = \partial B(a,r) \backslash \{w_0\}$. We assume that the orientation of γ from z_0 to z_1 is consistent with the positive orientation of ∂V. Then $f(\gamma)$ describes $\partial B(a,r) \backslash \{w_0\}$ oriented counter-clockwise. By our assumption, $f^{-1}(D)$ is connected, so we can connect z_1 with z_0 by a curve $\gamma' \subset f^{-1}(D)$, oriented from z_1 to z_0. So we obtain a closed curve $\gamma + \gamma'$. Let Γ be its image. As $f(\gamma')$ is contained in a simply connected domain D, and $a \in \partial D$, there exists a simple curve Σ from a to infinity which is disjoint from $f(\gamma')$. We may assume that this curve Σ intersects the circle $\partial B(a,r)$ once, at a point w_1. Then also $\Sigma \cap \Gamma = \{w_1\}$. We conclude that

$$\operatorname{ind}_a \Gamma = 1. \tag{2}$$

Here $\operatorname{ind}_a \Gamma$ denotes the winding number of the closed curve Γ with respect to the point a. Furthermore, using Lemma 1 we may assume that the branch ϕ of the inverse f^{-1} such that $\phi(w_1) \in \gamma$ has an analytic continuation along Σ in both directions, towards a and towards ∞. The preimage of Σ under this branch ϕ is a simple curve σ from b to infinity. Evidently, σ does not intersect γ' and has a one-point intersection with γ. We conclude that

$$\operatorname{ind}_b(\gamma \cup \gamma') = 1. \tag{3}$$

Now (3) implies that $\operatorname{ind}_a \Gamma \geq m$ and this contradicts (2). This contradiction completes the proof in this case that a is a critical value.

Now we consider the case that f^{-1} has a direct singularity over a. Let $r > 0$ such that a component V of $f^{-1}(B(a,r))$ contains no a-points of f. By Lemma 1, we can continue a branch of the inverse, mapping a point $w_0 \in B(a,r) \cap D$ to a point $z_0 \in V$, along a Jordan curve τ around a, with $\tau \subset B(a,r)$. Thus there exists a simple curve γ in V from z_0 to some point $z_1 \in V$ such that $\tau = f(\gamma)$ is a Jordan curve having a in its interior. Consider a curve γ' in $f^{-1}(D)$ that begins at z_1 and ends at z_0. Let Γ be the image of the closed curve $\gamma + \gamma'$. As before, we can construct a simple curve Σ (parametrized by $(-1,1)$) with the following properties: $\Sigma(t) \to a$ as $t \to -1$, $\Sigma(t) \to \infty$ as $t \to 1$, and moreover, $\Sigma \cap f(\gamma') = \emptyset$, and $\Sigma \cap f(\gamma)$ consists of one point $w' = \Sigma(0)$, where the curves Σ and $f(\gamma)$ cross each other, and this point w' has a simple preimage $z' \in \gamma$. In addition, using again Lemma 1, we may assume that the branch of f^{-1} that sends w' to z' has analytic continuation along Σ in both directions of Σ. Then the preimage σ of Σ passing through z' is a curve that tends to infinity in both directions. On the other hand this curve σ crosses the closed curve $\gamma + \gamma'$ exactly once, which is a contradiction. This completes the proof of part a).

For the proof of part b), we need some preliminaries. Our Lemmas 2 and 3 below are similar to the results of Shimizu [9, p. 186] and Terasaka [10, Lemma on p. 310]. Let f be an entire function. A curve $\gamma : [0,1] \to \mathbb{C}$ will be called *good for* f, if the set $\gamma([0,1])$ contains no critical values of f and all components of the full preimage $f^{-1}(\gamma([0,1]))$ are compact. To prove the existence of good curves we need the following version of the classical theorem of Gross [8, p. 292]:

Lemma 2. *Let ϕ be a holomorphic branch of the inverse f^{-1} defined in some disc B, and let ℓ be some direction in the plane. Then ϕ has an analytic continuation along almost all straight lines intersecting B and having the direction ℓ.*

Such lines can be parametrized by the points of their intersection with the diameter of B perpendicular to ℓ. "Almost all" refers to the Lebesgue measure on this diameter.

Proof of Lemma 2. We assume for simplicity that the direction ℓ is parallel to the real axis, and that the diameter of B perpendicular to this direction is (ia, ib), where $a < b$. Let $M > b - a$. Consider the rectangle

$$Q_M := \{z = x + iy : |x| < M, y \in (a,b)\}.$$

For each horizontal interval $\{x + iy_0 : |x| < M\}$, where $y_0 \in (a,b)$, we consider the maximal open subinterval containing the point iy_0 such that an analytic continuation of ϕ is possible along this subinterval. The union of these maximal subintervals over all $y_0 \in (a,b)$ forms a region $G_M \subset Q_M$. If a maximal horizontal interval in G_M has an endpoint inside Q_M, then we will call this endpoint a *singular point of* ϕ. It is enough to show that the Lebesgue measure of the projection of the set of singular points on the imaginary axis is zero, for every fixed $M > b-a$. The analytic continuation of ϕ along maximal horizontal intervals in G_M maps G_M univalently onto some region $G'_M \subset \mathbb{C}$. The singular points in G_M correspond to the critical values of f and to the accessible points at infinity of G'_M. Since the set of critical values is countable, the Lebesgue measure of its projection on the imaginary axis is zero. Let σ'_r be the intersection of G'_M with the circle $\{z : |z| = r\}$. We may assume that ϕ is bounded in B. Then for $r > r_0$ the set $\sigma_r := f(\sigma'_r)$ is a union of cross-cuts in G_M which separate the diameter $[ia, ib]$ from the set of singular points of ϕ on the boundary of G_M. It is enough to show that the length of σ_r tends to zero as $r \to \infty$ on some sequence.

We have

$$\operatorname{length}(\sigma_r) = \int_{\sigma'_r} |f'(z)|\,|dz|$$

and by Schwarz's inequality

$$\operatorname{length}^2(\sigma_r) \leq 2\pi r \int_{\sigma'_r} |f'(z)|^2\,|dz|.$$

Dividing by r and integrating with respect to r from r_0 to ∞, we obtain

$$\int_{r_0}^{\infty} \operatorname{length}^2(\sigma_r)\frac{dr}{r} \leq 2\pi \iint_{G'_M} |f'(z)|^2\,dxdy = 2\pi\,\operatorname{area}(G_M) \leq 2\pi\,\operatorname{area}(Q_M).$$

Thus the integral on the left hand side converges.

We conclude that $\operatorname{length}(\sigma_r) \to 0$ on some sequence $r = r_k \to \infty$. This proves the lemma.

Lemma 3. *Let f be an entire function, and $Q = (a, b, c, d)$ a rectangle in the plane. Then almost every closed interval connecting the opposite sides $[a, b]$ and $[c, d]$ and parallel to the other sides is good for f.*

Proof. Consider the set $\{B_j\}$ of all discs contained in Q, having rational center and rational radius; the discs are enumerated in arbitrary order. Applying Lemma 2 to the disc B_j and the direction $[a, d]$ we obtain an exceptional set of lines E_j of measure zero. Then $E = \bigcup E_j$ is a set of measure zero, and all intervals which are intersections of Q with lines parallel to $[a, d]$ and not in E are good for f. This proves the lemma.

Proof of Part b) of the Theorem. We will construct a curve Γ parametrized by $[0, 1)$ such that $\Gamma(t) \to \infty$ as $t \to 1$, and $\Gamma(0) = a$, and such that for every $\varepsilon \in (0, 1)$ the piece of the curve $\{\Gamma(t) : 0 \leq t \leq 1 - \varepsilon\}$ is good for f, and the unbounded component D of $\mathbb{C}\backslash\Gamma$ is not empty and unique.

Then it is easy to see that D has connected preimage, and $a \notin D$.

Our construction gives a piecewise linear curve Γ.

We begin with a curve Γ^0 (for example, a straight line segment) parametrized by $[1/2,3/4]$ which is good for f. The existence of such a curve follows from Lemma 3.

Let $\{\gamma_j^0\}$ be all components of the preimage $f^{-1}(\Gamma^0)$, with a fixed enumeration by positive integers. We choose a parametrization of each γ_j^0 which is consistent with the parametrization of Γ^0, that is $\Gamma^0(t)=f(\gamma_j^0(t)),\ 1/2\leq t\leq 3/4$.

Let $r_0>|\Gamma^0(1/2)-a|$. Consider the component U_0 of the preimage $f^{-1}(B(a,r_0))$ such that U_0 contains $\gamma_1^0(1/2)$. Our assumption about the point a implies that there exist $\rho_0\in(0,r_0/2)$ and a point z_0 in U_0 such that $f(z_0)=a$ and $f'(z_0)\neq 0$, and there is a branch ϕ_0 of f^{-1} in $B(a,\rho_0)$ that sends a to z_0. We may also assume that ϕ_0 is bounded in $B(a,\rho_0)$. Let σ_0 be a curve in U_0 from $\gamma_1^0(1/2)$ to z_0. Using Lemma 3, we can replace $f(\sigma_0)$ by a piecewise-linear curve $\Sigma^1:[1/4,1/2]\to B(a,r_0)$ which is good, which begins at $\Gamma^0(1/2)$ and ends at some point w_0 in $B(a,\rho_0)$. We define a new curve

$$\Gamma^1(t):=\begin{cases}\Gamma^0(t), & 1/2\leq t\leq 3/4,\\ \Sigma^1(t), & 1/4\leq t\leq 1/2.\end{cases}$$

Then Γ^1 is good, and the component of $f^{-1}(\Gamma^1)$ that contains γ_1^0 is compact.

The following fact is important: *no matter how we extend Γ^1 by attaching a piece in $B(a,\rho_0)$, parametrized by $[0,1/4]$, the component of the preimage of the extended curve that contains γ_1^0 will be compact,* because the part added to γ_1^0 will be contained in $\phi_0(B(a,\rho_0))$.

Now we repeat this process of extension. Suppose that Γ^n is parametrized by $[2^{-n-1},3/4]$. This curve is good for f and $\Gamma^n(2^{-n-1})\in B(a,\rho_{n-1})$. Let $\{\gamma_j^n\}$ be the component of $f^{-1}(\Gamma^n)$ that contains $\{\gamma_j^0\}$ such that the following property holds:

No matter how we extend Γ^n by adding a piece parametrized by the interval $[2^{-n-2},2^{-n-1}]$ within the disc $B(a,\rho_{n-1})$, the components of the full preimage of the extended curve which contain $\gamma_1^n,\dots,\gamma_n^n$ will be compact.

Now we perform the $(n+1)$-st extension step. Choose r_n satisfying

$$|\Gamma^n(2^{-n-1})-a|<r_n<\rho_{n-1}$$

and consider the component U_n of the preimage $f^{-1}(B(a,r_n))$ that contains $\gamma_{n+1}^n(2^{-n-1})$. Our assumption about the point a implies that U_n contains a simple preimage z_n of the point a, and that there exists $\rho_n\in(0,r_n/2)$ and a holomorphic branch ϕ_n of f^{-1} in $B(a,\rho_n)$ sending a to z_n. Let σ_n be a curve in U_n from $\gamma_{n+1}(2^{-n-1})$ to z_n. Using Lemma 3, we can replace $f(\sigma_n)$ by a piecewise-linear curve $\Sigma^{n+1}:[2^{-n-2},2^{-n-1}]\to B(a,r_n)$ which is good, which begins at $\Gamma^n(2^{n-1})$ and ends at some point w_n in $B(a,\rho_n)$. We define a new curve

$$\Gamma^{n+1}(t):=\begin{cases}\Gamma^n(t), & 2^{-n-1}\leq t\leq 3/4,\\ \Sigma^{n+1}(t), & 2^{-n-2}\leq t\leq 2^{-n-1}.\end{cases}$$

Then Γ^{n+1} is good, and for $j = 1, \dots, n+1$, the components of $f^{-1}(\Gamma^{n+1})$ that contain γ_j^0 are compact.

Finally we set $\Gamma^\infty(t) := \Gamma^n(t)$ for $t \in [2^{-n-1}, 2^{-n}]$ and $n = 1, 2, \dots$, and $\Gamma^\infty(t) = \Gamma^0(t)$ for $1/2 \le n \le 3/4$. It remains to extend the curve Γ^∞ with a piece parametrized by $[3/4, 1)$ such that $\Gamma(t) \to \infty$ as $t \to 1$, which is easy to do using Lemma 3.

This completes the proof of our theorem.

3. An example

We show that the function f given by (1) has infinitely many direct but no logarithmic singularity over 0. Let

$$g(z) := \sum_{k=1}^{\infty} \left(\frac{z}{2^k}\right)^{2^k}$$

so that $f(z) = \exp g(z)$. We fix ε with $0 < \varepsilon \le \frac{1}{8}$ and put $r_n := (1+\varepsilon)2^{n+1}$ and $r_n' := (1-2\varepsilon)2^{n+2}$ for $n \in \mathbb{N}$. For $j \in \{0, 1, \dots, 2^n - 1\}$ we define the sets

$$A_{j,n} := \left\{ r \exp\left(\frac{2\pi i j}{2^n}\right) : r \ge r_n \right\}, \quad i = \sqrt{-1},$$

$$B_{j,n} := \left\{ r \exp\left(\frac{\pi i}{2^n} + \frac{2\pi i j}{2^n}\right) : r_n \le r \le r_n' \right\},$$

and

$$C_{j,n}^{\pm} := \left\{ r \exp\left(\frac{\pi i}{2^n} + \frac{2\pi i j}{2^n} \pm \frac{r - r_n'}{r_{n+1} - r_n'} \frac{\pi i}{2^{n+1}}\right) : r_n' \le r \le r_{n+1} \right\}.$$

We shall show that if n is large enough, then

$$\operatorname{Re} g(z) > 2^{2^n} \quad \text{for} \quad z \in A_{j,n} \tag{4}$$

while

$$\operatorname{Re} g(z) < -2^{2^n} \quad \text{for} \quad z \in B_{j,n} \cup C_{j,n}^{+} \cup C_{j,n}^{-}. \tag{5}$$

Note that $C_{j,n}^{-}$ connects $B_{j,n}$ to $B_{2j,n+1}$ while $C_{j,n}^{+}$ connects $B_{j,n}$ to $B_{2j+1,n+1}$. This implies that

$$T := [-ir_1, ir_1] \cup \bigcup_{n=1}^{\infty} \bigcup_{j=0}^{2^n-1} \left(B_{j,n} \cup C_{j,n}^{+} \cup C_{j,n}^{-}\right)$$

is an infinite binary tree; see Figure 1. By (5), every unbounded simple path on this tree starting at 0 is an asymptotic curve on which $\operatorname{Re} g(z) \to -\infty$. Choosing U_ρ as the component of

$$\{z : \operatorname{Re} g(z) < \log \rho\} = \{z : |f(z)| < \rho\}$$

which contains the "tail" of this curve we thus obtain a transcendental singularity of f^{-1} over 0, and this singularity is direct because f has no zeros.

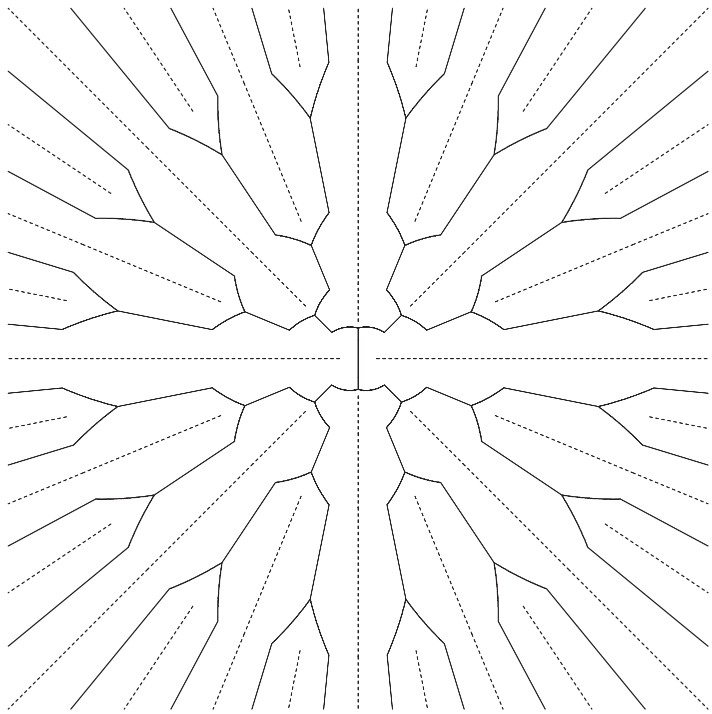

FIGURE 1. The part of the tree T lying in $\{z : |\operatorname{Re} z| \leq 80, |\operatorname{Im} z| \leq 80\}$, for $\varepsilon = 1/16$. The sets $A_{j,n}$ are drawn as dotted lines.

Using (4) we see that different curves define different singularities. Thus we obtain a set of direct singularities which has the power of the continuum.

Moreover, it follows from (4) and (5) and the above considerations that if U_ρ is a component of $\{z : |f(z)| < \rho\}$ containing the "tail" of some curve in T, then U_ρ also contains the "tail" of some other curve in T and thus there exists $\rho' < \rho$ such that U_ρ contains at least two components of $\{z : |f(z)| < \rho'\}$. This implies that the singularity defined by $\rho \mapsto U_\rho$ is not logarithmic.

To prove (4) we note that if $z = r\exp(2\pi i j/2^n) \in A_{j,n}$ so that $r \geq r_n$, then

$$\operatorname{Re} g(z) \geq \sum_{k=n}^{\infty} \left(\frac{r}{2^k}\right)^{2^k} - \sum_{k=1}^{n-1} \left(\frac{r}{2^k}\right)^{2^k} \geq \left(\frac{r}{2^n}\right)^{2^n} - \sum_{k=1}^{n-1} \left(\frac{r}{2^k}\right)^{2^k}.$$

Put $s := r/2^n$ and

$$\Sigma_1 := \sum_{k=1}^{n-1} \left(\frac{r}{2^k}\right)^{2^k}.$$

Then

$$\Sigma_1 = \sum_{k=1}^{n-1} \left(s2^{n-k}\right)^{2^k} \leq s^{2^{n-1}} \sum_{k=1}^{n-1} 2^{(n-k)2^k}.$$

Now $(n-k)2^k \leq 2^{n-1}$ for $1 \leq k \leq n-1$ and $s < 2 + 2\varepsilon$ so that

$$\Sigma_1 \leq s^{2^{n-1}}(n-1)2^{2^{n-1}} = o\left(s^{2^n}\right)$$

as $n \to \infty$ and hence

$$\operatorname{Re} g(z) \geq \left(\frac{r}{2^n}\right)^{2^n} - \Sigma_1 = (1 - o(1))s^{2^n} > 2^{2^n}$$

for large n. To prove (5) for $z \in B_{j,n}$, let $z = r\exp\left(\pi i/2^n + 2\pi ij/2^n\right) \in B_{j,n}$, with $r_n \leq r \leq r_n'$. Then

$$\operatorname{Re} g(z) \leq -\left(\frac{r}{2^n}\right)^{2^n} + \Sigma_1 + \Sigma_2$$

with

$$\Sigma_2 := \sum_{k=n+1}^{\infty} \left(\frac{r}{2^k}\right)^{2^k} = \sum_{k=n+1}^{\infty} \left(s2^{n-k}\right)^{2^k}.$$

Thus

$$\Sigma_2 \leq \left(\frac{s}{2}\right)^{2^n} + \sum_{k=n+2}^{\infty} \left(\frac{s}{4}\right)^{2^k}.$$

Since $s/4 \leq 1 - 2\varepsilon$ we find that

$$\Sigma_2 = o\left(s^{2^n}\right)$$

as $n \to \infty$ and thus

$$\operatorname{Re} g(z) \leq -(1 - o(1))s^{2^n} < -2^{2^n}$$

for $z \in B_{j,n}$, provided n is sufficiently large.

Finally we prove (5) for $z \in C_{j,n}^+$. So let

$$\begin{aligned} z &= r\exp\left(\frac{\pi i}{2^n} + \frac{2\pi ij}{2^n} + \frac{r - r_n'}{r_{n+1} - r_n'}\frac{\pi i}{2^{n+1}}\right) \\ &= r\exp\left(\frac{\pi i}{2^n} + \frac{2\pi ij}{2^n} + \frac{s - 4(1-2\varepsilon)}{12\varepsilon}\frac{\pi i}{2^{n+1}}\right) \\ &\in C_{j,n}^+, \end{aligned}$$

with $r_n' \leq r \leq r_{n+1}$, so that $4(1-2\varepsilon) \leq s \leq 4(1+\varepsilon)$. We have

$$\operatorname{Re} g(z) \leq \operatorname{Re}\left(\frac{z}{2^n}\right)^{2^n} + \operatorname{Re}\left(\frac{z}{2^{n+1}}\right)^{2^{n+1}} + \Sigma_1 + \Sigma_3$$

with

$$\Sigma_3 := \sum_{k=n+2}^{\infty} \left(\frac{r}{2^k}\right)^{2^k} \leq \sum_{k=n+2}^{\infty} \left(\frac{s}{4}\right)^{2^k} = o(1)$$

since $s > 4$. Since $\Sigma_1 = o\left(s^{2^n}\right)$ we find that

$$\begin{aligned} \operatorname{Re} g(z) &\leq s^{2^n} \cos\left(\pi + \frac{s-4(1-2\varepsilon)}{12\varepsilon}\frac{\pi}{2}\right) \\ &\quad + \left(\frac{s}{2}\right)^{2^{n+1}} \cos\left(\frac{s-4(1-2\varepsilon)}{12\varepsilon}\pi\right) + o\left(s^{2^n}\right) \\ &= s^{2^n}\left(\cos\left(\pi + t\frac{\pi}{2}\right) + \left(\frac{s}{4}\right)^{2^n}\cos(t\pi) + o(1)\right) \end{aligned}$$

as $n \to \infty$, with $t := (s - 4(1-2\varepsilon))/12\varepsilon$. The range $4(1-2\varepsilon) \leq s \leq 4(1+\varepsilon)$ corresponds to $0 \leq t \leq 1$ and $s = 4(1-2\varepsilon) + 12\varepsilon t$. We define

$$\begin{aligned} h(t) &:= \cos\left(\pi + t\frac{\pi}{2}\right) + \left(\frac{s}{4}\right)^{2^n}\cos(t\pi) \\ &= \cos\left(\left(1 + \frac{t}{2}\right)\pi\right) + (1 - 2\varepsilon + 3\varepsilon t)^{2^n}\cos(t\pi). \end{aligned}$$

and put $\delta := -\cos(11\pi/8)/2 > 0$. For $0 \leq t \leq \frac{1}{2}$ we have

$$h(t) \leq \cos\left(\frac{5}{4}\pi\right) + \left(1 - \frac{\varepsilon}{2}\right)^{2^n} < -2\delta$$

if n is large enough. For $\frac{1}{2} \leq t \leq \frac{3}{4}$ we have $\cos(t\pi) < 0$ and thus

$$h(t) \leq \cos\left(\left(1 + \frac{t}{2}\right)\pi\right) \leq \cos\left(\frac{11}{8}\pi\right) = -2\delta.$$

Finally, for $\frac{3}{4} \leq t \leq 1$ we have $\cos\left(\left(1 + \frac{t}{2}\right)\pi\right) < 0$ and thus

$$h(t) \leq (1 - 2\varepsilon + 3\varepsilon t)^{2^n}\cos(t\pi) \leq \left(1 + \frac{1}{4}\right)^{2^n}\cos\left(\frac{3}{4}\pi\right) \leq -2\delta$$

if n is large. Overall we find that $h(t) \leq -2\delta$ for all t and thus

$$\operatorname{Re} g(z) \leq -\delta s^{2^n} < -2^{2^n}$$

for $z \in C^+_{j,n}$, provided n is large enough. The proof that

$$\operatorname{Re} g(z) < -2^{2^n}$$

for $z \in C^-_{j,n}$ is analogous. This completes the proof of (4) and (5). As already mentioned, this implies that every path going to ∞ in T corresponds to a direct singularity of f over 0 which is not logarithmic, and the set of such singularities has the power of the continuum. Also, we see that if $\rho \to U_\rho$ is a singularity over 0 such that $U_\rho \cap T \neq \emptyset$ for all $\rho > 0$, then this singularity is not logarithmic.

It remains to prove that there are no other singularities over 0. Suppose that $\rho \to U_\rho$ is a singularity over 0 such that $U_\rho \cap T = \emptyset$ for some $\rho > 0$. In order to obtain a contradiction we note that it follows as in the proof of (4) and (5) that if $r_n \leq |z| \leq r'_n$, then

$$g(z) = (1 + \eta(z)) \left(\frac{z}{2^n}\right)^{2^n}$$

where $\eta(z) \to 0$ as $n \to \infty$. For large n we thus have $|\eta(z)| \leq \varepsilon^2 \leq \frac{1}{2}$. Differentiating we obtain

$$\frac{g'(z)}{g(z)} - \frac{2^n}{z} = \frac{\eta'(z)}{1 + \eta(z)}.$$

For $(1 + 2\varepsilon)2^{n+1} \leq |z| \leq (1 - 3\varepsilon)2^{n+2}$ we thus have

$$\begin{aligned} \left|\frac{g'(z)}{g(z)} - \frac{2^n}{z}\right| &\leq 2|\eta'(z)| \\ &= 2\left|\frac{1}{2\pi i}\int_{|\zeta - z| = \varepsilon 2^{n+1}} \frac{\eta(\zeta)}{(\zeta - z)^2} d\zeta\right| \\ &\leq 2\frac{1}{\varepsilon 2^{n+1}} \max_{|\zeta - z| = \varepsilon 2^{n+1}} |\eta(z)| \\ &\leq \frac{\varepsilon}{2^n} \end{aligned}$$

and hence

$$\left|\frac{zg'(z)}{g(z)} - 2^n\right| \leq \frac{\varepsilon|z|}{2^n} \leq 4\varepsilon(1 - 3\varepsilon) < \frac{1}{2}.$$

We deduce that

$$\begin{aligned} \frac{d \arg g(re^{i\theta})}{d\theta} &= \operatorname{Im}\left(\frac{d \log g(re^{i\theta})}{d\theta}\right) \\ &= \operatorname{Re}\left(\frac{re^{i\theta} g'(re^{i\theta})}{g(re^{i\theta})}\right) \\ &\geq 2^n - \frac{1}{2} \\ &> 0 \end{aligned}$$

for $(1+2\varepsilon)2^{n+1} \leq r \leq (1-3\varepsilon)2^{n+2}$ and large n. We conclude that $\arg g(re^{i\theta})$ is an increasing function of θ, and it increases by $2^n 2\pi$ as θ increases by 2π. Choose n and r as above so large that the circle $\{z : |z| = r\}$ intersects U_ρ, that (4) and (5) hold and that $-2^{2^n} < \log \rho$. From the behavior of $\arg g(re^{i\theta})$ we deduce that the circle $\{z : |z| = r\}$ contains at most 2^n arcs where $\operatorname{Re} g(re^{i\theta}) < \log \rho$. On the other hand, for $j \in \{0, 1, \ldots, 2^n - 1\}$ the points $r \exp(\pi i/2^n + 2\pi i j/2^n)$ are contained in such an arc by (5), and each of them is contained in a different one by (4). Hence there are precisely 2^n such arcs and each one contains one of the points $r \exp(\pi i/2^n + 2\pi i j/2^n)$. Thus each such arc intersects some $B_{j,n}$ and hence T. In particular, $U_\rho \cap \{z : |z| = r\}$

intersects T, contradicting the assumption that $U_\rho \cap T = \emptyset$. This completes the proof that f has no logarithmic singularities over 0.

The set of direct singularities over 0 of the function f we just constructed has the power of continuum. The following proposition explains the reason for this:

Proposition. *Let f be an entire function whose inverse has a direct singularity S over some $a \in \mathbb{C}$. Then either S is logarithmic, or every neighborhood of S is also a neighborhood of other direct singularities over a.*

It follows that all direct singularities of entire functions of finite order over finite points are logarithmic. Indeed, by the Denjoy–Carleman–Ahlfors theorem [8, p. 313], such functions have only finitely many transcendental singularities. This corollary is not new though we could not locate a source.

Another corollary of our Proposition is this: If f is an entire function with a finite asymptotic value a and if all singularities of f^{-1} over a are direct but not logarithmic, then the set of direct singularities over a has the power of continuum, as in our example above.

Proof of the Proposition. Suppose that $U = U_r$ is a neighborhood of exactly one direct singularity over a finite point a, for some $r > 0$. By the Maximum Principle, U is simply connected. It is easy to see that the closure of U in the Riemann sphere is locally connected. So a conformal map $\phi : B(0,1) \to U$ extends to a continuous map from the unit disc to the Riemann sphere. The preimage of infinity under ϕ is a closed set E, which by a theorem of Beurling [11, p. 344] has zero logarithmic capacity. We consider the positive harmonic function

$$u(z) := \log \frac{r}{|f(\phi(z)) - a|}, \quad z \in B(0,1).$$

It has a representation

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi+0}^{\pi+0} \frac{1 - r^2}{1 - 2r\cos(\varphi - \theta) + r^2} d\chi(\varphi)$$

for some increasing function $\chi : \mathbb{R} \to \mathbb{R}$, and we have $\lim_{r \to 1} u(re^{i\theta}) = \infty$ if $e^{i\theta} \in E$ while $\lim_{r\to 1} u(re^{i\theta}) = 0$ if $e^{i\theta} \notin E$; see [11, p. 144]. It is easy to see that if E consists of only one point, then E is logarithmic. Suppose now that E contains two points $z_1 = e^{i\theta_1}$ and $z_2 = e^{i\theta_2}$. As E is nowhere dense, there exists a simple cross-cut σ in $B(0,1)$ that separates z_1 and z_2 such that u is bounded on σ and both ends of the cross-cut belong to the complement of E. This crosscut divides the unit disc into two regions G_1 and G_2, and since $\lim_{r\to 1} u(re^{i\theta_j}) = \infty$ we see that u is unbounded in G_j, for $j = 1, 2$. The image $\phi(\sigma)$ of our cross-cut separates U into two regions $D_j = \phi(G_j)$. The harmonic function

$$v(z) := u(\phi^{-1}(z)) = \log \frac{r}{|f(z) - a|}$$

is unbounded in each D_j and bounded on ∂D_j for $j = 1, 2$. Thus there exists $\varepsilon > 0$ such that

$$\left\{ z \in U : v(z) > \log \frac{r}{\varepsilon} \right\} = \{ z \in U : |f(z) - a| < \varepsilon \}$$

is disconnected. As $f(z) \neq a$ for $z \in U$ we conclude that U is a neighborhood of at least two singularities over 0. This completes the proof of the Proposition.

Remark. It is much easier to find meromorphic functions with a direct singularity which is not logarithmic. For example, $f(z) = 1/(z \sin z)$ has two direct singularities over 0, but none of them is logarithmic, since their neighborhoods are multiply connected.

References

[1] I. N. Baker, Completely invariant domains of entire functions, in *Mathematical Essays Dedicated to A. J. Macintyre*, edited by H. Shankar, Ohio University Press, Athens, Ohio, 1970, 33–35.

[2] ———, The domains of normality of an entire function. *Ann. Acad. Sci. Fenn. (Ser. A, I. Math.)* 1 (1975), 277–283.

[3] W. Bergweiler, Iteration of meromorphic functions. *Bull. Amer. Math. Soc.* 29 (1993), 151–188.

[4] ———, A question of Eremenko and Lyubich concerning completely invariant domains and indirect singularities. *Proc. Amer. Math. Soc.* 130 (2002), 3231–3236.

[5] A. Eremenko, Geometric theory of meromorphic functions, in "In the tradition of Ahlfors and Bers, III." *Contemp. Math.* 355, Amer. Math. Soc., Providence, RI, 2004, pp. 221–230.

[6] A. E. Eremenko and M. Yu. Lyubich, Dynamical properties of some classes of entire functions. *Ann. Inst. Fourier* 42 (1992), 989–1020.

[7] F. Iversen, *Recherches sur les fonctions inverses des fonctions méromorphes.* Thèse, Helsingfors, 1914.

[8] R. Nevanlinna, *Eindeutige analytische Funktionen*, Springer, Berlin, Göttingen, Heidelberg, 1953.

[9] T. Shimizu, On the fundamental domains and the groups for meromorphic functions. I. *Jap. J. Math.* 8 (1931), 175–236.

[10] H. Terasaka, On the division of Riemann surfaces into sheets. *Jap. J. Math.* 8 (1931), 309–326.

[11] M. Tsuji, *Potential theory in modern function theory*, Maruzen, Tokyo, 1959; reprint by Chelsea, New York, 1975.

Mathematisches Seminar, Christian–Albrechts–Universität zu Kiel, Ludewig–Meyn–Str. 4, D–24098 Kiel, Germany

E-mail address: bergweiler@math.uni-kiel.de

Department of Mathematics, Purdue University, West Lafayette, IN 47907, USA

E-mail address: eremenko@math.purdue.edu