Dirichlet problem

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In this text we consider examples of solution of Dirichlet problem. See also "Poisson's formula" and "Uniqueness of solution of Dirichlet problem".

I recall the general statement:

Given a region D, and a function ϕ defined on the boundary ∂D, find a harmonic function u in D such that

$$\lim_{z \to \zeta} u(z) = \phi(\zeta), \quad \zeta \in \partial D.$$

We will make the following assumptions: function ϕ is bounded and continuous at all points of ∂D except finitely many, and look only for bounded solutions u. Under these conditions, solution is unique, if exists.

1. Solution for the upper half-plane H and piecewise constant ϕ. Notice that $\operatorname{Arg} z$ is a bounded harmonic function in H, and has boundary values 0 for $x > 0$ and π for $x < 0$. So it solves the Dirichlet problem with these boundary values. Using this function, we can solve the Dirichlet problem for H with any piecewise constant boundary function.

Example 1. Solve the Dirichlet problem for H with this boundary function:

$$\phi(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

Solution:

$$u(z) = 1 - \frac{1}{\pi} \operatorname{Arg} z.$$

Example 2. Solve the Dirichlet problem for H with this boundary function:

$$\phi(x) = \begin{cases} 0, & x < -1, \\ 2, & -1 < x < 1, \\ 0, & x > 1. \end{cases}$$

Solution:

$$u(z) = \frac{2}{\pi} \left(\mathrm{Arg}\,(z-1) - \mathrm{Arg}\,(z+1) \right).$$

Example 3. Let $a_0 < a_1 < \ldots < a_n$ and

$$\phi(x) = c_j, \quad \text{for} \quad a_{j-1} < x < a_j, \; 1 \leq j \leq n,$$

and $\phi(x) = 0$ for $x < a_0$ and $x > a_n$. Solution:

$$u(z) = \frac{1}{\pi} \sum_{j=1}^{n} c_j \left(\mathrm{Arg}\,(z - a_{j-1}) - \mathrm{Arg}\,(z - a_j) \right).$$

Indeed, when $a_{j-1} < x < a_j$ only one term in this sum is different from zero, and this term equals c_j. For $x < a_0$ and $x > a_n$, all terms are equal to zero.

The same method can be used to solve the

2. Dirichlet problem for the unit disk with piecewise-constant boundary function. Suppose that $a_1 = e^{i\theta_1}$ and $a_2 = e^{i\theta_2}$, are two points on the boundary and $0 <\leq \theta_1 < \theta_2 < \theta_1 + 2\pi$. Denote by $[a, b]$ the arc of the circle between a and b:

$$[a, b] = \{e^{i\theta} : \theta_1 < \theta < \theta_2\}.$$

Let z be a point in the unit disk. Connect it with straight segments with a and b and consider the angle between these segments at z. Of the two possible angles choose the angle *from* the vector $a - z$ *to* the vector $(b - z)$ counterclockwise. This angle equals

$$\mathrm{Arg}\,_0(b - z) - \mathrm{Arg}\,_0(a - z).$$

Make a picture! Check that this function is well defined in the unit disk. Since any branch of the argument is harmonic, this angle is a harmonic function in the unit disk. By elementary geometry, the boundary values are $\theta_2 - \theta_1)/2$ outside the arc $[a, b]$ and $\pi - (\theta_2 - \theta_1)/2$ inside this arc. This permits to solve the Dirichlet problem for the unit disk with piecewise constant boundary function.

Example 4. Solve the Dirichlet problem in the unit disk with the boundary function $\phi(e^{i\theta}) = 1$ for $0 < \theta < \pi/2$ and 0 on the rest of the circle.

Solution.

$$u(z) = \frac{1}{\pi}\left(\mathrm{Arg}_0(i-z) - \mathrm{Arg}_0(1-z) - \pi/4\right). \tag{1}$$

Remark. Let us plug $z = 0$ and check this result against the average property:

$$\frac{1}{\pi}\left(\mathrm{Arg}_0(i) - \mathrm{Arg}_0(1) - \pi/4\right) = 1/4.$$

Another form of the solution of Dirichlet problem for the disk was explained in the lectures and in the book. Tis is Poisson integral. For Example 4, this integral is

$$u(z) = \frac{1}{2\pi}\int_0^{\pi/2} \frac{1-r^2}{1+2r\cos(\theta-t)+r^2}, \quad \text{where} \quad z = re^{i\theta}.$$

So formula (1) actually computes this integral! Can you show directly that (**??**) and (1) is the same function?

3. Dirichlet problem for other regions. Here the following consideration is used. Let D be a region and f a one-to-one analytic function mapping D onto the upper half-plane H. If we know f and can solve the Dirichlet problem for H, then we can solve it for D. This is because for a harmonic function u is H, the function $u(f(z))$ is harmonic in D.

We know many functions which perform one-to-one analytic mappings.

a) $f(z) = az + b, \quad a \neq 0$ is one-to one in the whole plane. The map that it performs is a similarity combined with a shift. Using this function we can map any half-plane onto any other half-plane, any strip onto any other strip, any disk onto any other disk.

b) $f(z) = e^z$ is one-to-one in the strip $\{x + iy : 0 < y < \pi\}$ and maps it onto the upper half-plane H.

c) $f(z) = z^\alpha$, $\alpha > 1/2$, the principal branch is one-to-one in the sector

$$\{z : 0 < \mathrm{Arg}\, z < \pi/\alpha\}$$

onto the upper half-plane.

d) $J(z) = (z + 1/z)/2$ is one-to-one in these regions and maps them to:

Upper half-plane onto the plane with cuts $(-\infty, -1]$ and $[1, +\infty)$.

Lower half-plane onto the same.

Unit disk onto the plane with a cut $[-1, 1]$.

Outside of the unit disk to the same.

e) $\sin z$ maps the half-strip $\{x + iy : -pi/2 < x < \pi/2, y > 0\}$ onto the upper half-plane. Combining with a), this permits to map any half-strip (with interior angles $\pi/2$) onto H.

Example 5. Consider the half-strip

$$S = \{x + iy : x > 0, |y| < 1\}.$$

Solve the Dirichlet problem for S with boundary function $\phi(z) = 1$ on the interval $(-i, i)$ and zero on the rest of the boundary.

Solution. We know that $\sin z$ maps the half-strip

$$S_1 = \{x + iy : -\pi/2 < x/\pi/2,\ y > 0\}$$

onto the upper half-plane H. To use this fact, we first find a map of the strip S onto S_1. This map must be of the form $g(z) = az + b$, and we find that $a = i\pi/2,\ b = 0$ will do the job.

Our interval $(-i, i)$ is mapped by $g(z)$ onto the interval $(-pi/2, \pi/2)$ and then by $\sin z$ to the interval $(-1, 1)$.

Solution of the Dirichlet problem for the half-plane H with boundary values 1 on $(-1, 1)$ and 0 on the rest is obtained similarly to Example 2, it is

$$\frac{1}{\pi}\left(\mathrm{Arg}\,(z - 1) - \mathrm{Arg}\,(z + 1)\right).$$

So the solution of the original problem is

$$\frac{1}{\pi}\left(\mathrm{Arg}\ (\sin(\pi iz/2) - 1) - \mathrm{Arg}\ (\sin(\pi iz/2) + 1)\right).$$