An ellipse can be described by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

A parametrization is obtained if we put

$$x = a\cos t, \quad y = b\sin t, \quad 0 \le t \le 2\pi.$$

The formula for the length of a curve from Calculus gives the length as

$$\int_0^{2\pi} \sqrt{(x')^2 + (y')^2} dt = \int_0^{2\pi} \sqrt{a^2 \sin^2 t + b^2 \cos^2 t} dt.$$

We may assume without loss of generality that $b \ge a$. The expression under the integral can be transformed as

$$\sqrt{b^2 - (b^2 - a^2)\sin^2 t} = b\sqrt{1 - e^2 \sin^2 t},$$

where $e = \sqrt{b^2 - a^2}/b$ is called the eccentricity. The quantities b (the length of the larger semi-axis) and the eccentricity describe the size and the shape of the ellipse.

It is sufficient to find the length of the arc of the ellipse in the first quadrant, because the ellipse consists of four such arcs of equal lengths.

Thus we need to evaluate the integral

$$\int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 t} dt,$$

which gives the length of 1/4 of the ellipse whose larger semi-axis is 1.

There is no closed form expression using only elementary functions and constants related to them, like e and π. However there is the following series development:

$$\int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 \theta}\, d\theta = \frac{\pi}{2}\left(1 - \frac{1}{2\cdot 2}e^2 - \frac{1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4}e^4 - \frac{1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}e^6 - \dots\right)$$

$$= \frac{\pi}{2}\left(1 - \frac{1}{2\cdot 2}e^2 - \sum_{n=2}^{\infty} \frac{(2n-1)!!(2n-3)!!}{((2n)!!)^2} e^{2n}\right).$$

Sketch of the proof. First, denote $e^2 \sin^2 \theta$ by x and use the binomial formula:

$$(1-x)^{1/2} = 1 - \frac{1}{2}x - \sum_{n=2}^{\infty} \frac{(2n-3)!!}{2^n n!} x^n.$$

Second,

$$\int_0^{\pi/2} \sin^{2n}\theta\, d\theta = \frac{\pi}{2} 2^{-2n} \binom{2n}{n}.$$

The last integral is computed by the residues. This is one of our standard integrals.