ON THE ZEROS OF MEROMORPHIC FUNCTIONS OF THE FORM $f(z) = \sum_{k=1}^{\infty} \frac{a_k}{z-z_k}$

By

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Abstract. We study the zero distribution of meromorphic functions of the form $f(z) = \sum_{k=1}^{\infty} \frac{a_k}{z-z_k}$ where $a_k > 0$. Noting that f is the complex conjugate of the gradient of a logarithmic potential, our results have application in the study of the equilibrium points of such a potential.

Furthermore, answering a question of Hayman, we also show that the derivative of a meromorphic function of order at most one, minimal type has infinitely many zeros.

1. Introduction

Consider a meromorphic function

$$f(z) = \sum_{k=1}^{\infty} \frac{a_k}{z - z_k}, \quad a_k > 0. \tag{1.1}$$

We suppose that

$$\sum_{k=1}^{\infty} \frac{a_k}{|z_k|} < \infty \tag{1.2}$$

and thus that the series in (1.1) converges absolutely for all $z \in \mathbf{C}$, $z \neq z_k$. The function f is the complex conjugate to the gradient of the logarithmic potential

$$u(z) = \sum_{k=1}^{\infty} a_k \log \left| 1 - \frac{z}{z_k} \right|, \tag{1.3}$$

which is a subharmonic function of order at most one, convergence class. This follows from (1.2). The zeros of f are the equilibrium or critical points of u. If the

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a_k are all positive integers, we may also consider the entire function

$$F(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_k}\right)^{a_k}.$$

In this case $f = F'/F$.

The zeros of f were studied in [3] where the following results were obtained:

Theorem 1.1 *If all the a_k are positive integers, then f has infinitely many zeros.*

Theorem 1.2 *If all the a_j are positive real numbers and*

$$\sum_{\{j:|z_j|\leq r\}} a_j = o(\sqrt{r}), \quad r \longrightarrow \infty, \tag{1.4}$$

then f has an infinite set of zeros.

The condition (1.4) means that the function u given by (1.3) has order at most 1/2, minimal type.

The paper [3] also contains results for the case when the a_k are real but not necessarily positive, as well as some counterexamples. Some earlier results on the distribution of zeros of f, with complex a_k are contained in [6, Ch. V].

Throughout this paper we use the standard notation of value distribution theory for meromorphic and subharmonic functions. The reader is referred to [6], [8], [9] and [10]. Our first result extends Theorem 1.1 to allow for non-integer values of a_k.

Theorem 1.3 *If f defined by* (1.1) *has order at most one, minimal type, then f has an infinite set of zeros.*

We remark that, in general, there is no relationship between the growth of f and u. However, if we assume that the a_k are bounded away from zero, then $T(r,f) = O(N(r,u))$ and we obtain the following corollary, whose proof is immediate from (1.3).

Corollary 1.4 *Suppose that $a_k \geq a > 0$ in* (1.1). *Then f has an infinite set of zeros.*

Thus Theorem 1.3 is stronger than Theorem 1.1.

In [3] it was proved that if F is a meromorphic function of order less than 1/2, then F'/F has infinitely many zeros. Examples were given to show that there exist merormorphic functions of any order greater than or equal to 1/2 whose log derivatives have no zeros. This led Hayman to ask whether the derivative of a meromorphic function of order less than one has any zeros. Our next theorem answers this question affirmatively.

Theorem 1.5 *Let F be a transcendental meromorphic function of order at most one, minimal type; then F' has infinitely many zeros.*

For F of order at most one convergence class, it is easily seen that Theorem 1.5 is equivalent to the following

Theorem 1.6 *If f is as in* (1.1) *and the a_k are integers, with $a_k \geq -1$ for $j > j_0$, then f has infinitely many zeros.*

To establish the equivalence, note that under the conditions of Theorem 1.6 we have $f = g'/g$, where g is a meromorphic function with at most finitely many multiple poles, having order at most one, convergence class. Applying Theorem 1.5 to $F = 1/g$ we conclude that $f = g'/g$ has infinitely many zeros. Conversely if F is a meromorphic function of order at most one convergence class, then either F has infinitely many multiple zeros, or else we may apply Theorem 1.6 to $f = -F'/F$. In either case Theorem 1.5 is true.

We mention that Theorem 1.5 is sharp. Indeed for $\rho \geq 1$, it was shown in [3] that there exists an entire function G of order ρ such that G'/G has no zeros. Then $F = 1/G$ is meromorphic of order ρ and F' has no zeros.

The next two theorems give quantitative information on the distribution of the zeros of f in certain cases.

Theorem 1.7 *If the function f defined in (1.1) has lower order $\lambda < 1$ and*

$$0 < a \leq a_k \leq A < \infty \tag{1.5}$$

for some constants a and A, then $\delta(0,f) < 1$.

Corollary 1.8 *Let F be an entire function of order $\rho < 1$. Further suppose that the multiplicity of the zeros of F is uniformly bounded. Then $\delta(0, F'/F) < 1$.*

Theorem 1.9 *If f defined by* (1.1) *has lower order $\lambda < 1/2$, then*

$$\delta(0,f) \leq 1 - \cos \pi\lambda.$$

Further if $\lambda = 1/2$, then

$$\delta(0,f) < 1.$$

Corollary 1.10 *For entire functions F of order $\rho < 1/2$, we have*

$$\delta(0, F'/F) \leq 1 - \cos \pi\lambda.$$

Further if $\lambda = 1/2$, then

$$\delta(0, F'/F) < 1.$$

We note that the proofs of the corollaries follow immediately from their respective theorems.

Remark There is a conjecture of W. H. J. Fuchs which states that for entire functions F of lower order $\lambda < 1/2$, $\delta(0, F'/F) = 0$. Our Corollary (1.10) proves this if $\lambda = 0$.

We propose the following related problem:

Problem *Let $\rho < 1/2$ be the order of the subharmonic function u in* (1.3). *Can one estimate $\delta(0,f)$ in terms of ρ?*

We note that if the a_k are bounded from below then $\lambda(f) \leq \rho(u) < 1/2$ and Theorem 1.9 gives an answer.

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2. Proof of Theorems 1.3 and 1.5

We need the following

Proposition 2.1 *Let $\epsilon > 0$ and let g be a transcendental meromorphic function, of order at most one, minimal type having only finitely many poles. Let Γ be a path such that $\Gamma(t) \to \infty$ as $t \to \infty$ and*

$$\frac{\log |g(z)|}{\log |z|} \longrightarrow \infty \quad as \quad z \longrightarrow \infty, \quad z \in \Gamma. \tag{2.1}$$

Then there exists a domain S with the following properties:

a. *If $\theta(t) = \operatorname{meas}\{\theta \in [0, 2\pi] : te^{i\theta} \in S\}$, then for some $r_0 > 0$, we have*

$$\phi_0(r) := \log r - \pi \int_{r_0}^{r} \frac{dt}{t\theta(t)} \to +\infty$$

as $r \to +\infty$.

b. *For some $r_1 > 0$ the part of Γ lying in $\{z : |z| \geq r_1\}$ is contained in S.*

c. *For any $z_1, z_2 \in S$ there exists a path γ from z_1 to z_2 satisfying*

$$\int_{\gamma} |g(z)|^{-1} |dz| < \epsilon.$$

We postpone the proof of Proposition 2.1 to the end of this section.

Proof of Theorem 1.3 We now suppose that f has only finitely many zeros, and we set $g = 1/f$. Then by a theorem of Lewis, Rossi and Weitsman [12], there is a path Γ such that (2.1) holds, and such that

$$\int_\Gamma |g(z)|^{-1}|dz| < \infty \tag{2.2}$$

(The result was stated in [12] for entire functions, but the required generalization is trivial.) Applying Proposition 2.1 with $\epsilon = 1$, we see from (2.2), **b.**, **c.** and the fact that $|g(z)|^{-1} \doteq |\text{grad } u(z)|$, that $u(z) \le C$ on S, for some positive constant C. Let

$$\theta_1(t) = \text{meas}\{\theta \in [0, 2\pi] : u(te^{i\theta}) > C\}$$

so that

$$\theta_0(t) + \theta_1(t) \le 2\pi \tag{2.3}$$

where $\theta_0(t) = \theta(t)$ is as in **a.**.

Using [14, p.116], **a.** and the fact that u has at most order one, minimal type, we have for some $t_0 > 0$, and for $j = 0, 1$ that

$$\phi_j(r) := \log r - \pi \int_{t_0}^r dt/(t\theta_j(t)) \longrightarrow +\infty$$

as $r \to \infty$. Setting $\phi(r) = \min\{\phi_0(r), \phi_1(r)\}$, we have

$$\log^2(r/t_0) = \left(\int_{t_0}^r \frac{dt}{t}\right)^2 \le \int_{t_0}^r \frac{\theta_j(t)dt}{t} \cdot \int_{t_0}^r \frac{dt}{t\theta_j(t)} \le \pi^{-1}(\log r - \phi(r)) \int_{t_0}^r \frac{\theta_j(t)dt}{t}.$$

Adding these inequalities for $j = 0, 1$ and using (2.3), we obtain

$$\log^2(r/t_0) \le (\log r - \phi(r)) \log r,$$

which is a contradiction since $\phi(r) \to +\infty$. □

Proof of Theorem 1.5 Assuming that F' has only finitely many zeros, we set $g = 1/F'$. Applying the result of Lewis, Rossi and Weitsman [12] again, we obtain a path Γ such that (2.1) and (2.2) hold. Since $g = 1/F'$, we conclude from (2.2) that F tends to a finite value as $z \to \infty$ on Γ. We may assume that this value is 0.

But F/F' also has at most order one, minimal type, and has at most finitely many poles. Moreover, F/F' must be transcendental by the growth restrictions on F. It follows [12] that there exists a path Γ_1 such that

$$\log|F(z)/F'(z)|/\log|z| \longrightarrow +\infty \quad as \quad z \in \Gamma_1 \longrightarrow \infty,$$

and

$$\int_{\Gamma_1} |F'(z)/F(z)||dz| < +\infty.$$

We now see that F must tend to a finite, nonzero value, which we may assume to be 1, as $z \to \infty$ on Γ_1. Thus g grows transcendentally on Γ_1.

We now assume without loss of generality that $|F(z)| < 1/4$ for all $z \in \Gamma$ and $|F(z) - 1| < 1/4$ for all $z \in \Gamma_1$. Applying Proposition 2.1 again with $\epsilon = 1/4$, we obtain $r_0 > 0$ and domains S_0 and S_1 with the following properties. For $j = 0, 1$, we have $|F(z) - j| < 1/2$ for all $z \in S_j$, and for $\theta_j(t) := \text{meas}\{\theta \in [0, 2\pi] : te^{i\theta} \in S_j\}$, we have

$$\theta_0(t) + \theta_1(t) \leq 2\pi.$$

This follows since S_0 and S_1 are disjoint. Also for $j = 0, 1$,

$$\phi_j(r) := \log r - \pi \int_{r_0}^{r} dt/(t\theta_j(t)) \longrightarrow +\infty.$$

We now obtain a contradiction exactly as in the proof of Theorem 1.3.

Before proving Proposition 2.1, we need the following lemma. (Compare with [15, p.44].)

Lemma 2.2 *There exists a monotone increasing sequence $R_n \in (2^{2n-2}, 2^{2n})$ such that for large n the total length of the level curves $|g(z)| = R_n$ in*

$$\mathcal{D}_n = \{z : |z| < 2^n\}$$

is at most $2^{3n/2}$.

Proof For $R > 0$ we have

$$n(2^n, Re^{i\theta}, g) \leq N(2^{n+2}, 1/(g - Re^{i\theta})) \leq T(2^{n+2}, g) + \log^+ R + O(1)$$

Thus

$$(2.4) \qquad p_n(R) := \frac{1}{2\pi} \int_0^{2\pi} n(2^n, Re^{i\theta}, g)d\theta \leq T(2^{n+2}, g) + \log^+ R + O(1).$$

Let $l_n(R)$ denote the total length of the level curves $|g(z)| = R$ in $\mathcal{D}_n$. Put $\beta_n = 2^{2n}, \alpha_n = 2^{2n-2}$. By the length – area principle (c. f. [7, p. 18]), we have

$$\int_{\alpha_n}^{\beta_n} \frac{l_n{}^2(R)dR}{Rp_n(R)} \leq 2\pi \cdot \text{area}(\mathcal{D}_n) = 2\pi^2 \cdot 2^{2n}.$$

So there exists $R_n \in (\alpha_n, \beta_n)$ such that

$$l_n^2(R_n) \le \frac{1}{\beta_n - \alpha_n} R_n p_n(R_n) \cdot 2\pi^2 \cdot 2^{2n}.$$

From (2.4) and the fact that g has order at most one, minimal type, we have that for n large enough $p_n(R_n) \le o(2^n)$. This and the obvious fact that the sequence $\{R_n\}$ is monotone increasing proves the lemma. □

Proof of Proposition 2.1 To prove Proposition 2.1, we take a sequence R_n as in Lemma 2.2, noting that we are free to choose the R_n so that the level curves $|g(z)| = R_n$ have no multiple points, and are never tangent to any of the circles $\{z : |z| = 2^n\}$.

We take $P > 0$ so that all the poles of g lie in $\{z : |z| < P\}$, and for n so large that $R_n > M(P, g)$, we set

$$U_n = \{z : P < |z| < 2^n, \quad |g(z)| > R_n\} \tag{2.5}$$

We set $U = \bigcup_{n=n_0}^{\infty} U_n$, where n_0 is so large as to satisfy certain conditions to be specified later. Now if $m \ge n_0$ and n_0 is sufficiently large, the part of Γ lying in $\{z : 2^{m-1} \le |z| < 2^m\}$ is contained in U_m, by (2.1), and we define S to be the component of U which contains the part of Γ lying in $\{z : |z| \ge 2^{n_0-1}\}$. Thus **b.** is trivially satisfied.

Now suppose that $z_0 \in \partial S$. Then for some n, z_0 lies on the boundary of some component of U_n. Therefore, either $|z_0| < 2^n$ and $|g(z_0)| = R_n$ or $|z_0| = 2^n$ and $|g(z_0)| \ge R_n$. In the latter case we must have $|g(z_0)| \le R_{n+1}$ for otherwise z_0 is an interior point of some component of U_{n+1} and so is an interior point of S, since S is a component of U. Moreover, if $|z_0| < 2^n$ and $|g(z_0)| = R_n$ with $n > n_0$, then we must have that $|z_0| \ge 2^{n-1}$, for otherwise z_0 is in U_{n-1} and so is an interior point of S. Since U_{n_0} is bounded away from zero, there exists therefore a positive constant L so that

$$|g(z)| \le L|z|^2, \quad z \in \partial S. \tag{2.6}$$

Now consider the function

$$w(z) = \log|g(z)| - 2\log|z| - \log L$$

which is subharmonic in $\{z : |z| > P\}$. By (2.1) and (2.6), S contains an unbounded component S' of the set $\{z : w(z) > 0\}$. Setting

$$\theta'(t) = \text{meas}\{\theta \in [0, 2\pi] : te^{i\theta} \in S'\},$$

we have [14, p.116] for some positive t_1 that

$$\log r - \pi \int_{t_1}^{r} dt/(t\theta'(t)) \longrightarrow +\infty,$$

which proves **a.**, since $S' \subset S$.

To prove **c.**, we recall that ∂S consists of some arcs of the level curves $|g(z)| = R_n$ lying in $\{z : |z| < 2^n\}$, together with some arcs of the circles $\{z : |z| = 2^n\}$ on each of which $R_n \leq |g(z)| \leq R_{n+1}$. Moreover ∂S has no multiple points, and each component of ∂S is a piecewise analytic, simple curve. Now using Lemma 2.2, we have

$$\int_{\partial S} |g(z)|^{-1}|dz| \leq \sum_{n_0}^{\infty} R_n^{-1}(2^{3n/2} + 2\pi 2^n) < \epsilon/2 \tag{2.7}$$

if n_0 is chosen large enough. Now given z_1 and z_2 in S, we note that the straight line segment from z_1 to z_2 meets ∂S finitely often. If w_k, w_{k+1} are two such intersection points such that the open line segment joining them lies in a component V of $\mathbf{C} \setminus \overline{S}$, then ∂V must have a bounded subarc ω joining w_k to w_{k+1}, and we replace the line segment between w_k and w_{k+1} by ω. This gives a path from z_1 to z_2 through $\overline{S}$, which we can easily replace by a simple path γ. Now if T_n is the part of γ which lies on the straight line segment between z_1 and z_2 and lies in $\{z : 2^{n-1} \leq |z| < 2^n\}$ (or in $\{z : P < |z| < 2^n\}$ if $n = n_0$), then

$$\int_{T_n} |g(z)|^{-1}|dz| < 2^{n+1}/R_n < 2^{3-n}$$

so that using 2.7, we have **c.** provided n_0 is chosen large enough. □

3. Proof of Theorem 1.7

By a theorem of M. Keldysh–I. V. Ostrovski (cf. [6, Ch. V, Theorem 6.1] we have

$$m(r,f) = o(1), \qquad r \longrightarrow \infty, \tag{3.1}$$

and so

$$T(r,f) = N(r,f) + O(1), \qquad r \longrightarrow \infty. \tag{3.2}$$

Choose a sequence $r_k \to \infty$ of strong Pólya peaks of order λ for

$$N(r,u) := \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})d\theta$$

(cf. [13]). Then

$$N(r,u) \leq (1 + o(1))\left(\frac{r}{r_k}\right)^{\lambda} N(r_k,u), \quad {A_k}^{-1}r_k \leq r \leq A_k r_k, \quad A_k \to \infty, \tag{3.3}$$

and

$$T(r,u) \le C\left(\frac{r}{r_k}\right)^{\lambda} N(r_k,u) \quad A_k{}^{-1}r_k \le r \le A_k r_k. \tag{3.4}$$

From the condition (1.5), Jensen's formula and (3.2) we conclude that for sufficiently large r

$$\frac{a}{2}T(r,f) \le N(r,u) \le 2AT(r,f), \tag{3.5}$$

so

$$T(r,f) \le C_1\left(\frac{r}{r_k}\right)^{\lambda} T(r_k,f), \quad A_k{}^{-1}r_k \le r \le A_k r_k, \quad A_k \to \infty. \tag{3.6}$$

Consider the sequence of δ- subharmonic functions

$$w_k(z) = \frac{\log|f(r_k z)|}{T(r_k,f)}.$$

Using (3.6) and a theorem of Anderson and Baernstein [1, Theorem 4 and Theorem 5], we conclude that the sequence w_k is *normal*. That is, we may choose a subsequence, also denoted w_k, such that

$$w_k(z) \longrightarrow w(z), \qquad k \longrightarrow \infty. \tag{3.7}$$

Here the convergence in (3.7) holds in $L^1_{loc}(dxdy)$ and the convergence holds in $L^1(d\theta)$ for any circle $\{re^{i\theta} : 0 \le \theta \le 2\pi\}$. Furthermore the Riesz mass (generalized Laplacian) of w_k converges weakly to that of w. Finally [2] the convergence holds in *1-measure* in $\mathbf{C}$, that is, given $\epsilon > 0$, and K compact, the set

$$\{z : |w_k(z) - w(z)| \ge \epsilon\} \cap K$$

can be covered by the union of disks, the sum of whose radii approaches zero as k approaches infinity (see also [5]).

From (3.1) and the L^1 convergence on circles, we have that

$$w(z) \le 0, \qquad z \in \mathbf{C}. \tag{3.8}$$

Suppose that

$$\delta(0,f) = 1.$$

This implies that

$$N(r, 1/f) = o(T(r,f)), \qquad r \longrightarrow \infty, \tag{3.9}$$

so that, for suitably adjusted A_k

$$N(r, 1/f) = o(T(r_k,f)), \qquad r/r_k \le A_k, \qquad k \longrightarrow \infty.$$

Thus by the weak convergence of the Riesz mass of w_k to that of w, we have that w has no positive mass and is hence superharmonic.

By (3.9) $m(r, 1/f) = (1 + o(1))T(r,f)$ as $r \to \infty$ and hence

$$\frac{1}{2\pi}\int_0^{2\pi} w(e^{i\theta})d\theta = -1.$$

In particular

(3.10) $$w \not\equiv 0.$$

Consider the subharmonic function u defined in (1.3). It follows from (3.4) that the sequence

$$v_k(z) = \frac{u(r_k z)}{N(r_k, u)}$$

will again be normal in the sense of Anderson and Baernstein and after passing to a subsequence we have $v_k \to v$ where v is a subharmonic function having order at most λ. Here the convergence is to be interpreted as in (3.7). Also $v_k, v \not\equiv 0$ since

$$\frac{1}{2\pi}\int_0^{2\pi} v(re^{i\theta})d\theta = 1.$$

Denote by E a component of the set $\{z : w(z) \leq -1\}$, which contains a point z such that $w(z) < -1$. (Hayman and Kennedy [9] call such components "thick".) Then E is closed because w is lower semicontinuous. The minimum principle implies that E is unbounded. As the order of w is less then one, there is exactly one thick component [9]. To complete the proof of Theorem 1.7 we need the following lemma.

Lemma 3.1 *The function $v(z)$ is constant on E.*

Once Lemma 3.1 is proved, we proceed as follows. Set $u_1 = -w - 1$. Then u_1 is subharmonic and $u_1 \leq 0$ on $\mathbf{C} \setminus E$. Further v is subharmonic and $v(z) \equiv c$ on E. So the function $h := (v - c)^+ + u_1{}^+$ is subharmonic, has order $\lambda < 1$ and the set $\{z : h(z) \geq 0\}$ has at least two thick components. This contradicts Theorem 8.9 in [10].

We conclude the proof of Theorem 1.7 by proving Lemma 3.1.

Proof of Lemma 3.1 Fix a point $z_0 \in E$ and $\epsilon > 0$. We know that w is superharmonic and $w(z_0) \leq -1$. By the Wiener criterion [10, Ch. 7.1], there is a set $X \subset (0, \epsilon)$ of positive linear measure such that

$$r \in X \Longrightarrow w(z_0 + re^{i\theta}) \leq -1/2, \qquad |\theta| \leq \pi.$$

Since w_k converges to w in 1-measure, we can find a circle C centered at z_0 of arbitrarily small radius such that

i. $w(z) \leq -1/2$, for $z \in C$.

ii. $w_k \to w$ uniformly on C for an appropriate subsequence still denoted w_k (see [5]).

Now note that

$$\bar{f} = \text{grad}\, u. \tag{3.11}$$

Set $C_k = \{r_k\zeta : \zeta \in C\}$ and recall the definition of w_k. We obtain by (3.11), **i.** and **ii.** that

$$|\text{grad}\, u(z)| \leq \exp(-T(r_k,f)/4)$$

for $z \in C_k$ and for $k \geq k_0$, where k_0 depends only on C. This gives immediately that for $z \in C_k$ and $k \geq k_0$

$$\text{oscillation}_{z \in C_k} u \leq r_k \text{diam}(C) \exp(-T(r_k,f)/4).$$

Since the right hand side of the above inequality approaches zero as k approaches infinity, we obtain

$$\text{oscillation}_{z \in C} v_k \longrightarrow 0, \qquad k \longrightarrow \infty.$$

and conclude that v is constant on C.

Thus every point $z_0 \in E$ may be surrounded by arbitrarily small circles on which v is constant. Since E is closed and connected, we may cover E by open disks $\{D_k\}_{k=1}^{\infty}$ such that v is constant on ∂D_k for each k, each compact subset of E intersects only a finite number of disks, and $Y = \bigcup_{k=1}^{\infty} \partial D_k$ is a connected set. Thus $v(z) \equiv c$ on Y for some constant c. By the maximum principle, $v(z) \equiv c$ on $\bigcup_{k=1}^{\infty} D_k \supset E$. The proof of Lemma 3.1 and hence of Theorem 1.7 is complete. □

4. Proof of Theorem 1.9; $\lambda < 1/2$

Suppose that $\lambda < 1/2$ and that

$$\delta(0,f) = 1 - c, \qquad c < \cos \pi\lambda. \tag{4.1}$$

Then by a theorem of Goldberg [6, Ch. 5, Theorem 3.2],

$$\liminf_{r \to \infty} \frac{\log M(r,f)}{T(r,f)} < 0,$$

where $M(r,f) = \max_{|z|=r}\{|f(z)|\}$ In particular

$$M(r_k,f) \leq r_k^{-2} \tag{4.2}$$

for some sequence $r_k \to \infty$.

We need the following lemma.

Lemma 4.1 *Let f be as in* (1.1). *Then*

$$\max_{|z|=r}\{|f(z)|\} \geq c/r^{-1}$$

where $r > 0$ and c is a positive constant.

Note that once we prove Lemma 4.1, it and (4.2) lead to a contradiction. Hence (4.1) must be false and Theorem 1.9 must be true.

Proof of Lemma 4.1 Let u be as in (1.3) and denote $B(r) = \max_{|z|=r}\{u(z)\}$. Fix $r_0 > 0$ and let $z_0 = r_0e^{i\theta}$ be a point such that $u(z) = B(r_0)$. Then if $r < r_0$, we have

$$u(r_0e^{i\theta}) - u(re^{i\theta}) \geq B(r_0) - B(r),$$

and hence

$$\frac{\partial u}{\partial r}(z_0) \geq \frac{dB}{dr}(r_0-).$$

Thus

$$|\text{grad } u(z_0)| \geq \frac{dB}{dr}(r_0-).$$

The lemma follows from this, (3.11) and the fact that the maximum of a subharmonic function is a convex increasing function of $\log r$ [9, p. 66]. □

5. Proof of Theorem 1.9; $\lambda = 1/2$

For the sake of exposition we assume throughout that $\lambda = 1/2$ is the order of f. The main tools used here are theorems in [4] and [11] both of which have explicitly stated lower order analogues (in sections 9 and V respectively). We leave the details to the interested reader.

We may write

$$f = f_1/f_2, \tag{5.1}$$

where f_1 and f_2 are entire functions with no common zeros, both having orders no greater than 1/2. By (3.1) and (5.1) we obtain

$$T(r,f) = N(r, 1/f_2) + o(1)$$

and

$$N(r, 1/f) = N(r, 1/f_1).$$

So if we assume that

$$\delta(0,f) = 1,$$

then

$$N(r, 1/f_1) = o(N(r, 1/f_2)), \qquad r \to \infty. \tag{5.2}$$

Rotate the zeros of f_1 and f_2 to the negative axis and form the respective canonical products F_1 and F_2. For g entire, define

$$m_0(r, g) = \min_{|z|=r} |g(z)|.$$

Classically [10, §6.1.1] for $i = 1, 2$,

$$\log m_0(r, F_i) \le \log m_0(r, f_i) \le \log M(r, f_i) \le \log M(r, F_i), \tag{5.3}$$

$$\log m_0(r, F_i) + \log M(r, F_i) \le \log m_0(r, f_i) + \log M(r, f_i) \tag{5.4}$$

and

$$\log M(r, F_i) = r \int_0^\infty \frac{N(t, 1/f_i)}{(r+t)^2} dt. \tag{5.5}$$

We obtain from (5.2), (5.3), and (5.5) that

$$\log M(r, f_1) \le o(\log M(r, F_2)), \quad r \to \infty. \tag{5.6}$$

From Lemma 4.1 and (5.1) it follows that

$$M(r, f_1)/m_0(r, f_2) \ge M(r, f) \ge cr^{-1} \tag{5.7}$$

Now (5.7), (5.6) and (5.3) imply that

$$\log m_0(r, f_2) \le o(\log M(r, F_2)), \quad r \to \infty, \tag{5.8}$$

$$\log m_0(r, F_2) \le o(\log M(r, F_2)), \quad r \to \infty. \tag{5.9}$$

We use the following result of Drasin and Shea [4] which concerns the case of equality in the $\cos \pi\lambda$-theorem:

If an entire function F_2 of order $1/2$ satisfies (5.9) then there exists a set $E \subset [0, \infty]$ of logarithmic density 1 such that for $r \in E$, $r \to \infty$

$$\log M(r, F_2) = r^{1/2} L(r), \tag{5.10}$$

where L is a slowly varying function in the sense of Karamata on E (cf. [4, p. 233]),

$$N(r, F_2) = \left(\frac{2}{\pi} + o(1)\right) \log M(r, F_2) \tag{5.11}$$

and

$$|\log m_0(r, F_2)| = o(\log M(r, F_2)). \tag{5.12}$$

(Recall that the logarithmic density of a set E is defined by

$$\lim_{r\to\infty} \frac{\int_{E\cap[1,r]} dt/t}{\log r}.$$

If the limit does not exist we may define upper and lower logarithmic density in the obvious way.)

Now it follows from (5.10), (5.3), (5.4), (5.12) and (5.8) that

$$\text{(5.13)} \qquad \begin{aligned} \log M(r,f_2) &= (1+o(1))\log M(r,F_2) \\ &= (1+o(1))r^{1/2}L(r), \quad r\in E, \quad r\to\infty. \end{aligned}$$

As $N(r,f_2)\equiv N(r,F_2)$, (5.11) implies that

$$\text{(5.14)} \qquad N(r,f_2) = \left(\frac{2}{\pi}+o(1)\right) r^{1/2}L(r).$$

Then (5.13), (5.14) and [4, Lemma 8.1] imply that for every $\delta > 0$ there exist $\epsilon > 0$, a set $E_1 \subset E$ of logarithmic density one and subsets $K(r) \subset [0,2\pi]$ with $\text{meas}(K(r)) < \delta$ such that

$$\text{(5.15)} \qquad \log|f_2(re^{i\theta})| \geq \epsilon \log M(r,f_2), \quad \theta\in[0,2\pi]\backslash K(r), \quad r\in E_1.$$

Now from (5.6) and (5.13) it follows that

$$\text{(5.16)} \qquad \log M(r,f_1) = o(\log M(r,f_2)), \quad r\in E, \quad r\to\infty$$

and (5.15), (5.16) and (5.1) imply

$$\int_{[0,2\pi]\backslash K(r)} r|f(re^{i\theta})|d\theta \to 0, \quad r\in E_1, r\to\infty.$$

To get a contradiction we apply the following

Lemma 5.1 *Let f be as in* (1.1). *Then there exists a $\delta > 0$ and a set E_0 of positive lower logarithmic density such that if $K(r)$ is any set of angular measure no greater than δ then*

$$\frac{1}{2\pi}\int_{[0,2\pi]\backslash K(r)} r|f(re^{i\theta})|d\theta \to \infty,$$

when $r\to\infty$ on E_0.

6. Proof of Lemma 5.1

When f is the logarithmic derivative of an entire function F of finite order, Lemma 5.1 is contained, but not explicitly stated in [11]. We follow their arguments extremely closely.

Let u be the subharmonic function given by (1.3). Define

$$T(r) := \frac{1}{2\pi}\int_0^{2\pi} u^+(re^{i\theta})d\theta, \quad A(r) := \frac{dT(r)}{d\log r} \quad \text{and} \quad n(r) := \sum_{\{k:|z_k|\le r\}} a_k.$$

Differentiation under the integral gives

$$A(r) = \frac{1}{2\pi}\int_{\{\theta:u(re^{i\theta})>0\}} \operatorname{Re}(re^{i\theta}f(re^{i\theta}))d\theta,$$

while by the Residue Theorem we have

$$n(r) = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}(re^{i\theta}f(re^{i\theta}))d\theta.$$

So

$$\max\{A(r), n(r)\} \le \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}^+(re^{i\theta}f(re^{i\theta}))d\theta. \tag{6.1}$$

Suppose that the lemma is false. Then, for any set E_0 of positive lower logarithmic density, there are subsets $J(r) \subset [0, 2\pi]$ such that $\operatorname{meas}(J(r)) \to 0$ and

$$\int_{[0,2\pi]\setminus J(r)} r|f(re^{i\theta})|d\theta = O(1), \tag{6.2}$$

where $r \to \infty, r \in E_0$.

We find in exactly the same way as in [11, (4.17), (4.19)-(4.21)], that

$$\begin{aligned} \int_{J(r)} \operatorname{Re}^+(re^{i\theta}f(re^{i\theta}))d\theta &\le n(r) - n(r/e) + o(A(r) + n(r)) \\ &\le (\eta + o(1))n(r) + o(A(r)), \end{aligned} \tag{6.3}$$

as $r \to \infty$, on a set of positive lower logarithmic density, where $\eta \in (0, 1)$. To derive (6.3) we use Lemmas 2, 3 and 5 from [11] which are just growth lemmas for increasing functions and the differentiated Poisson–Jensen formula for u [11, (3.1)].

Then (6.3) and (6.2) contradict (6.1), and the lemma is proved. □

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