Докл. Акад. Наук СССР
Том 316 (1991), № 3

Soviet Math. Dokl.
Vol. 43 (1991), No. 1

ON THE DISTRIBUTION OF VALUES OF MEROMORPHIC FUNCTIONS OF FINITE ORDER

UDC 517.53

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Given a sequence of complex numbers $z_k \to \infty$, we define a counting measure κ by setting $\kappa(E) = \operatorname{card}\{k : z_k \in E\}$ for every $E \subset \mathbf{C}$.

Let f be a meromorphic function of finite order $\rho > 0$. We fix pairwise distinct $a_1, \dots, a_q \in \overline{\mathbf{C}}$, $q \geq 3$, and we denote by μ_k the counting measures of the sequences of a_k-points of the function f, taking account of multiplicity. We study the asymptotic behavior of the measures μ_k and the relations among them by using the methods of the theory of limit sets of subharmonic functions [1]–[5]. We employ the standard notation of Nevanlinna theory.

1. Let $V(r) = r^{\rho(r)}$, where $\rho(r)$ is a proximate order of the function f. There is a representation $f = f_1/f_2$, where the f_i are entire functions with the property

$$T(r, f_i) = O(V(r)), \qquad r \to \infty, \quad i = 1, 2 \tag{1}$$

(the f_i may vanish simultaneously). Let μ_0 be the counting measure for the sequence of common zeros of the functions f_1 and f_2. We consider the subharmonic functions

$$u_k = \log|f_1 - a_k f_2| \tag{2}$$

(if $a_k = \infty$, then $u_k = \log|f_2|$). The Riesz measure of the function u_k equals $\mu_k + \mu_0$. We introduce another subharmonic function:

$$u = \log\sqrt{|f_1|^2 + |f_2|^2}, \tag{3}$$

the Riesz measure of which has the form $\mu + \mu_0$, where μ is an absolutely continuous measure with density $\pi^{-1}|f'|^2/(1+|f|^2)^2$. Then

$$T(r, f) = \int_0^r \frac{dt}{t}\,\mu(\{z : |z| \leq t\}),$$
$$N(r, a_k, f) = \int_1^r \frac{dt}{t}\,\mu_k(\{z : |z| \leq t\}) + O(\log r), \qquad r \to \infty.$$

For each $r \geq 1$, we define the operator L_r acting on functions w and on measures κ by the formulas

$$(L_r w)(z) = w(rz)/V(r), \qquad (L_r\kappa)(E) = \kappa(rE)/V(r),$$

where $E \subset \mathbf{C}$ is an arbitrary Borel set. Furthermore, we employ the topology of the space D' of generalized functions—the dual to the space of infinitely differentiable, compactly supported functions. It follows from (1) that the families $(L_r u)$, $(L_r u_k)$, $(L_r\mu)$, and $(L_r\mu_k)$, $r \geq 1$, are precompact in D'. Let $r_j \to \infty$ be an arbitrary sequence, and let $L_j = L_{r_j}$. Choosing a subsequence, if necessary, we may assume that $L_j u_k \to v_k$, $1 \leq k \leq q$; $L_j\mu_k \to \nu_k$, $0 \leq k \leq q$; and $L_j u \to v$ and $L_j\mu \to \nu$,

1991 *Mathematics Subject Classification*. Primary 30D35.

$j \to \infty$; here v and v_k are subharmonic functions with Riesz measures $\nu + \nu_0$ and $\nu_k + \nu_0$ respectively. The limit measures ν_k characterize the asymptotic distribution of a_k-points of the function f, and the measure ν describes the distribution of a-points for almost all $a \in \bar{\mathbf{C}}$ [4].

The set of functions $w : \mathbf{C} \to \mathbf{R}$ is provided with a natural partial order: $w_1 \le w_2$ if $w_1(z) \le w_2(z)$, $z \in \mathbf{C}$. The space of charges in $\mathbf{C}$ (a charge is the difference of two locally finite Borel measures) is also provided with a partial order: $\kappa_1 \le \kappa_2$ if $\kappa_2 - \kappa_1$ is a measure. We denote the least upper bound and greatest lower bound of finite families of charges or functions (relative to the above ordering) by $\vee$ and $\wedge$ respectively.

It follows from (2) and (3) that $u(z) = u_j(z) \vee u_k(z) + O(1)$, $z \to \infty$, $1 \le j < k \le q$, whence

$$v = v_j \vee v_k, \qquad 1 \le j < k \le q. \tag{4}$$

In particular, we have $v = \bigvee_{1 \le k \le q} v_k$. Relation (4) was used in [5] to derive an inequality analogous to the second fundamental theorem of Nevanlinna theory. Here we derive a sharper relation.

We recall that the *fine topology* in $\mathbf{C}$ is the smallest topology in which all subharmonic functions are continuous [7]. We consider the fine open sets

$$E_k = \{z : v_k(z) < v(z)\}. \tag{5}$$

It follows from (4) that $E_j \cap E_k = \varnothing$ when $j \ne k$. We denote by ν_k^* the restriction of the charge ν_k to the set $\mathbf{C} \setminus E_k$.

THEOREM. *It follows from* (4) *that*

$$\sum_{k=1}^{q} (\nu - \nu_k^*) = 2\nu - \bigwedge_{1 \le k < j \le q} (\nu_k^* + \nu_j^*). \tag{6}$$

If f is entire, we take $a_q = \infty$. Then $\nu_q = 0$, and (6) gives

$$(q-2)\nu = \sum_{k=1}^{q-1} \nu_k^* - \bigwedge_{1 \le k \le q-1} \nu_k^*.$$

In particular, when $q = 3$ we obtain the relation $\nu = \nu_1^* \vee \nu_2^*$.

By weakening (6), we obtain that for meromorphic functions

$$\sum_{k=1}^{q} (\nu - \nu_k) \le 2\nu. \tag{7}$$

Hence we may deduce the second fundamental theorem of Nevanlinna theory in the form

$$\sum_{k=1}^{q} m(r, a_k, f) \le 2T(r, f) + o(V(r)), \qquad r \to \infty.$$

Details and refinements are contained in [5].

Relation (6) has two advantages in comparison with the second fundamental theorem. Firstly, (6) and (7) have a local character, which makes it possible to investigate Borel rays, filling discs, etc. [4]. Secondly, (6) is an equality, while the second fundamental theorem is an inequality. Our approach does not use the derivative. This is both an advantage (the possibility of generalizations [5]) and a disadvantage: (6) does not contain information on multiple points of the function f.

2. Auxiliary assertions from potential theory.

LEMMA 1 ([7], p. 186). *Let* w_1 *and* w_2 *be subharmonic functions with Riesz measures* κ_1 *and* κ_2. *If* $w_1(z) = w_2(z)$, $z \in E$, *then the restrictions of the measures* κ_1 *and* κ_2 *to the fine interior of the set* E *coincide.*

A function w that is representable as the difference of two subharmonic functions is called *δ-subharmonic.* We denote by $\mu[w]$ the Riesz charge of the δ-subharmonic function w. The operators $\wedge$ and $\vee$, applied to finite families, do not go out of the class of δ-subharmonic functions.

LEMMA 2. *Let* $w_1, \dots, w_n$ *be δ-subharmonic functions. Then*

$$\mu\left[\bigwedge_{k=1}^{n} w_k\right] \geq \bigwedge_{1\leq k<j\leq n} \mu[w_k \wedge w_j].$$

This assertion was proved in [5] for continuous functions $w_1, \dots, w_n$. B. Fuglede kindly explained to the authors that an analogous proof carries over to the general case if one uses results from fine potential theory [7]–[9].

We will say that a relation between charges holds on the set X if it is true for the restrictions of these charges to X.

LEMMA 3 [10]. *Let* w_1 *and* w_2 *be δ-subharmonic functions with* $w_1 \leq w_2$. *If* $E = \{z : w_1(z) = w_2(z)\}$, *then* $\mu[w_1] \leq \mu[w_2]$ *on* E.

3. Proof of the theorem. We may assume that $\nu_0 = 0$, for otherwise we subtract the potential of the measure ν_0 from all the functions v and v_k. The new functions satisfy (4) as before, and their Riesz measures will be ν and ν_k respectively. The sets E_k and the measures ν_k^* do not change.

Fixing k, we will verify (6) on E_k. By definition,

$$\nu_k^* = 0 \quad \text{on } E_k. \tag{8}$$

Furthermore, if $j \neq k$, then $v_j(z) = v(z)$ for $z \in E_k$ in view of (4). Inasmuch as the set E_k is fine open, we conclude by Lemma 1 that

$$\nu_j^* = \nu_j = \nu \quad \text{on } E_k \text{ if } j \neq k. \tag{9}$$

It follows from (8) and (9) that (6) holds on E_k.

We now prove (6) on $E = \mathbf{C} \setminus \bigcup E_k = \{z : v(z) = \bigwedge_{1\leq k\leq q} v_k(z)\}$. We set

$$w_j = v + v_j; \tag{10}$$

$$w = \bigwedge_{1\leq j\leq q} w_j = v + \bigwedge_{1\leq j\leq q} v_j = \sum_{j=1}^{q} v_j - (q-2)v \tag{11}$$

(the last equality holds by (4)). From (11) we obtain

$$\mu[w] = \sum_{j=1}^{q} \nu_j - (q-2)\nu.$$

In particular,

$$\mu[w] = 2\nu - \sum_{j=1}^{q} (\nu - \nu_j^*) \quad \text{on } E. \tag{12}$$

We estimate the restriction of the charge $\mu[w]$ from above and below. In view of (4), for all $k \neq j$ we have $w \leq v_k + v_j$, where there is equality on E. We conclude by Lemma 3 that

$$\mu[w] \leq \nu_k + \nu_j = \nu_k^* + \nu_j^* \quad \text{on } E,$$

or

$$\mu[w] \leq \bigwedge_{1 \leq k < j \leq q} (\nu_k^* + \nu_j^*) \quad \text{on } E. \tag{13}$$

On the other hand, for all j and k we set

$$w_{jk} = w_j \wedge w_k \geq v_j + v_k \tag{14}$$

and note that (14) reduces to equality on E. By Lemma 3,

$$\mu[w_{jk}] \geq \nu_j + \nu_k = \nu_j^* + \nu_k^* \quad \text{on } E.$$

Finally, applying Lemma 2, we obtain

$$\mu[w] \geq \bigwedge_{1 \leq k < j \leq q} \mu[w_{jk}] \geq \bigwedge_{1 \leq k < j \leq q} (\nu_j^* + \nu_k^*) \quad \text{on } E. \tag{15}$$

The relations (13) and (15) together with (10) give (6) on E. The theorem is proved.

The authors thank B. Fuglede for clarifying questions of fine potential theory.

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Received 15/MAY/90

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Translated by HAROLD P. BOAS