Erdős' problem on the length of lemniscates

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Let $p(z) = z^d + \ldots$ be a monic polynomial of degree d. Consider the level set $E(p) = \{z : |p(z)| = 1\}$. What is the maximal length of this set?

This goes back to the paper of Erdős, Herzog and Piranian, J. d'Analyse math, 1958. The conjectured extremal polynomial is of course $z^d + 1$, so the maximal length is supposed to be $2d + o(1)$, as $d \to \infty$. It is not hard to prove the estimate $O(d)$, and the best known explicit estimate is $9.173\,d$ (Eremenko and Hayman, Mich. J., 1999). We also proved that for extremal polynomials the level sets $E(p)$ have to be connected, which reduces the number of parameters of the problem by the factor of two. In particular, it follows that the conjecture is true for $d = 2$.

However, the question is open even for $d = 3$, and even for this case, I know about an attempt to obtain a rigorous computer-assisted proof, which failed.

Erdős repeated this problem many times in various problem lists. He offered \$100 prize first, and later this was raised to \$200. I don't know whether prizes promised by him are paid after his death, but for this problem I will pay myself \$200 *for the first complete solution which I will receive and verify.*