The number of solutions of some non-holomorphic equations

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For a polynomial P of degree d, equation $P(z) = 0$ has at most d solutions. This equation can be written as a system of two real equations in two real variables x, y, where $z = x + iy$. To such systems Besout theorem applies and gives the upper estimate d^2 for the number of solutions. What special features of this real systems imply that the number of solutions is estimated as a linear function of the degree?

Consider an equation

$$P(\overline{z}, z) = 0, \tag{1}$$

where P is a polynomial of degree m in $\overline{z}$ and degree $n > m$ is $\overline{z}$. Besout theorem gives an upper estimate n^2 for the number of solutions.

Conjecture. *Equation (1) has at most*

$$C_1(m)n + C_1(m)$$

solutions, for some functions C_1, C_2.

There was a more precise conjecture of Wilmshurst, with the estimate $3n - 2 + m(m - 1)$, but this was disproved in [8].

An exact estimate is known for $m = 1$: in this case, the number of solutions is at most $5n-5$, and this is best possible. This result has important applications to astronomy (gravitational lensing), see [6]. It is proved by an application of (anti-)holomorphic dynamics, the argument invented by Khavinson and Swiatek [5], and used in all subsequent papers on the subject.

The result of Khavinson and Newman has been generalized to some transcendental equations of the form $z = f(\overline{z})$, where f is meromorphic in [1, 2, 7]. I mention one of these results [2]:

Equation

$$\zeta(z) + Az + B\overline{z} = 0 \tag{2}$$

where ζ is the Weierstrass' ζ-function, has at most 5 solutions in a period parallelogram. Here A and B are constants which make the LHS periodic with respect to the lattice of ζ.

Explicit expressions for A and B are the following:

$$A = \frac{\pi}{4\omega_1^2 \mathrm{Im}\,\tau} - \frac{\eta_1}{\omega_1}, \quad B = -\frac{\pi}{4|\omega_1|^2 \mathrm{Im}\,\tau},$$

using the standard notation of the theory of elliptic functions.

Equation (2) comes from a geometric problem of counting the metrics of constant positive curvature with one conic singularity with angle 6π on a torus [9, 4]. It also was used in the theory of minimal surfaces [10, 3].

It is desirable to find a generalization of these results to systems of equations. In particular, this system of of great interest:

$$A\sum_{j=1}^{n} z_j + B\sum_{j=1}^{n} \overline{z}_j + \sum_{j=1}^{n} \zeta(z_j) = 0,$$

$$\sum_{j \neq k} \left(\zeta(z_j - z_k) - \zeta(z_j) + \zeta(z_k)\right) = 0, \quad 1 \leq k \leq n.$$

Constants A and B are the same as in (2). This system describes metrics f constant positive curvature on tori with one conic singularity with angle $2\pi(2n+1)$.

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