

# How Euler found the sum of reciprocal squares

A. Eremenko

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In the lectures, the formula

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad (1)$$

was derived using residues. Euler found this in 1735, 90 years before Cauchy introduced residues. Even complex numbers were not commonly used in Euler's time.

So it is interesting and useful to see how Euler found this. His first contribution was a sophisticated numerical computation. He computed the sum to 20 decimal places. As an exercise, you can try to estimate, how many terms of the series are needed for this, assuming that you just add the terms.

Now I will explain Euler's proof. It is based on a new representation of the sine function

$$\sin z = z - \frac{z^3}{6} + \frac{z^5}{120} - \frac{z^7}{5040} + \dots \quad (2)$$

We know (and Euler knew) that this series converges for all values of  $z$ . In modern language we say that sine is an entire function. Euler's great idea was to treat an entire function as a polynomial of infinite degree.

We know that every polynomial factors:

$$P(z) = c(z - z_1)(z - z_2) \dots (z - z_n), \quad (3)$$

where  $z_k$  are the roots and  $c$  is a constant. Can we somehow extend this factorization to entire functions, that is to polynomials of infinite degree? An entire function can have infinitely many roots, for example  $\sin z$  has roots

$z_k = k\pi$ , where  $k$  is any integer. So one has to expect an infinite product in the factorization.

Which brings the question: *Can we make sense of infinite products?* So let us consider an infinite product of numbers

$$\prod_{n=1}^{\infty} a_n. \tag{4}$$

We can define partial products

$$p_N = \prod_{n=1}^N a_n,$$

and then define the infinite product as the limit of partial products when  $n \rightarrow \infty$ . If the limit exists, the product will be called convergent.

There is one little catch here: if one of the  $a_n$  equals zero, then all  $p_N$  with sufficiently large  $N$  will be zero, the limit evidently exists in this case, no matter what  $a_k$  with  $k > n$  are. This is not good, because we want to have a notion of convergence which does not depend of the finitely many terms of the product.

This and other considerations lead to the following definition:

*The product (4) is called convergent if for some  $m$  the partial products*

$$p_{m,N} = \prod_{n=m}^N$$

*have non-zero limit  $p$ , as  $N \rightarrow \infty$ . This number  $a_1 a_2 \dots a_{m-1} p$  is the value of the infinite product.*

So finitely many terms do not affect the convergence.

Now we have a simple *necessary* condition of convergence:  $a_n \rightarrow 1$ . Indeed, if  $p_{m,n}$  tends to a non-zero limit, then

$$a_n = \frac{p_{m,n+1}}{p_{m,n}} \rightarrow 1.$$

This is similar to the necessary condition of convergence of a series: the term of the series must tend to 0.

Now back to the product (3). Recall that the zeros of an entire function have no accumulation points in the plane, so  $z_k \rightarrow \infty$ . Is it possible then for the infinite product

$$(z - z_1)(z - z_2)(z - z_3) \dots$$

to converge? The answer is no, because  $(z - z_k) \rightarrow \infty$ .

However we can modify (3) in the following way:

$$c' \left(1 - \frac{z}{z_1}\right) \left(1 - \frac{z}{z_2}\right) \dots$$

What we did is just divided each factor on  $-z_k$ , and adjusted the constant  $c$  in front accordingly.

Now  $(1 - z/z_k) \rightarrow 1$  as  $k \rightarrow \infty$ , and at least the necessary condition of convergence will be satisfied.

As  $\sin 0 = 0$ , we have to put the factor  $z$  separately, and thus we conjecture that

$$\sin z = cz \prod_{n=-\infty, n \neq 0}^{\infty} \left(1 - \frac{z}{\pi n}\right). \quad (5)$$

There is a little difficulty here: we defined convergence and partial sums for a product with  $n$  from 1 to  $\infty$ , while in (5) we need a product over all integers  $n$ .

One can take the same approach as we had for Laurent series, and define convergence of such a product as a separate convergence of two products

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{\pi n}\right) \quad \text{and} \quad \prod_{n=1}^{\infty} \left(1 + \frac{z}{\pi n}\right).$$

But both these products diverge: the partial products of the first one tend to zero, and the partial products for the second one tend to infinity. For example, when  $z = 1$ , we have

$$\prod_{n=1}^N \left(1 + \frac{1}{\pi n}\right) > 1 + \sum_{n=1}^N \frac{1}{\pi n},$$

and the right hand side is a partial sum of a divergent series.

Euler's next idea is to group the terms with positive and negative  $n$ :

$$\sin z = cz \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\pi^2 n^2}\right).$$

As an exercise, try to prove that now the product in the right hand side converges.

It remains to find the value of  $c$ . But this is simple: divide both sides by  $z$ , and let  $z \rightarrow 0$ . Using that  $\sin z/z \rightarrow 1$  as  $z \rightarrow 0$ , we obtain

$$\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\pi^2 n^2}\right). \quad (6)$$

Now Euler proposes to actually perform the multiplication on the right hand side and compare with the power series (2). The term with  $z$  is certainly 1. The term with  $z^3$  is

$$-z \sum_{n=1}^{\infty} \frac{z^2}{\pi^2 n^2}.$$

Thus must be equal to  $-z^3/6$  in (2), so

$$\sum_{n=1}^{\infty} \frac{1}{\pi^2 n^2} = \frac{1}{6}.$$

So formula (6) implies the formula for the sum of reciprocal squares. This is the essence of Euler's first proof.

But did we really prove (6)?

Suppose that the convergence of the right hand side for all  $z \neq \pi n$  is justified. Then the right hand side is an entire function with the same roots as  $\sin$ , including multiplicity. Moreover, derivatives at 0 of the right and left hand side match. Can we conclude that (6) is valid?

Euler himself returned to this problem again and again during his life... He did many various checks of the formula (6) but I am not sure that any of his proofs satisfies the modern requirements of rigor. He also gave many alternative proofs of (1).

*Exercises.*

1. Prove

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}.$$

2. Prove

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}.$$

3. Prove

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

All these come from Euler's papers.

We will derive (6) from the partial fraction decomposition of  $\cot$ , also known to Euler.

$$\cot z = \frac{1}{z} + \sum_{k=-\infty, k \neq 0}^{\infty} \left( \frac{1}{z - k\pi} + \frac{1}{k\pi} \right). \quad (7)$$

To prove this by the method explained in the lectures, we choose as a contour  $C_n$  a disc or a square centered at the origin which contains integers from  $-n$  to  $n$ . Then, applying the residue theorem we obtain

$$\frac{1}{2\pi i} \int_{C_n} \frac{\cot \zeta}{\zeta - z} = \cot z - \frac{1}{z} - \sum_{k=-n, k \neq 0}^n \frac{1}{z - k\pi}. \quad (8)$$

Indeed the poles of the expression under the integral are:

- $\zeta = z$  with residue  $\cot z$ ,
- $\zeta = 0$  with residue  $-1/z$ , and
- $\zeta = \pi k$  with residues  $1/(\pi k - z)$ .

We would like to pass to the limit in (8) as  $n \rightarrow \infty$ . However this is not completely trivial: About  $\cot$  we only know that it is bounded on the contour, and this is not enough to show that the integral tends to 0. Also the resulting series in the RHS will be divergent in the usual sense.

One possible way to fix the problem is the following. Using the formula  $\cot z - 1/z \rightarrow 0$  as  $z \rightarrow 0$ , we pass to the limit when  $z \rightarrow 0$  in (8). We obtain

$$\frac{1}{2\pi i} \int_{C_n} \frac{\cot \zeta}{\zeta - z} d\zeta = \sum_{k=-n, k \neq 0}^n \frac{1}{k\pi}. \quad (9)$$

Now subtract (9) from (8). We obtain

$$\frac{z}{2\pi i} \int_{C_n} \frac{\cot \zeta}{\zeta(\zeta - z)} d\zeta = \cot z - \frac{1}{z} - \sum_{k=-n, k \neq 0}^n \left( \frac{1}{z - k\pi} + \frac{1}{k\pi} \right). \quad (10)$$

Now we can pass to the limit because the integral in the left hand side tends to 0 and the series obtained in the right hand side converges! For convergence of the series, notice that the expression in parentheses equals

$$\frac{z}{k\pi(z - k\pi)},$$

so for fixed  $z$  the absolute value is at most  $Ck^{-2}$ , and the series converges. So we obtain the expansion (7).

To derive (6), notice that

$$\cot z = \frac{(\sin z)'}{\sin z} = (\log \sin z)',$$

wherever  $\log \sin$  is defined. So we can integrate the series (7) along some path, and then take exp of the result. Notice that the integral is not single valued: every time we go around a pole  $2\pi i$  is added. However when we take exp of the result, all these  $2\pi i$  are killed, so the result is well defined. Choosing the beginning of the path of integration at 0, and taking  $1/z$  to the LHS when integrating, we obtain

$$\sin z = Cz \prod_{k=-\infty, k \neq 0}^{\infty} \left(1 - \frac{z}{\pi k}\right) e^{z/(\pi k)},$$

where  $C$  is the constant of integration. To find it we let  $z \rightarrow 0$  and use  $\sin z/z \rightarrow 1$ . This gives  $C = 1$ . Grouping the terms with  $k$  and  $-k$  we obtain (6).

Alternatively, we could group the terms with  $k$  and  $-k$  in (7), obtain

$$\cot z = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2 - (k\pi)^2}.$$

Then we could integrate and take exp to obtain (6).

One can also group the terms with positive and negative  $k$  in (8), and then prove that the integral tends to 0, however this last task is somewhat subtle: the reason why it tends to 0 is that the parts on the upper and lower half-circles almost cancel each other.