Math 425/525 Midterm exam, Fall 2020.

NAME:

1. Evaluate

\[ 2i \log \frac{1 - i}{1 + i}. \]

Your answer should in the form \( a + ib \) where \( a \) and \( b \) are real.
2. Describe and sketch the image of the sector

\[ \{ z : |z| < 1, \pi/2 < \text{Arg} \, z < \pi \} \]

under the Joukowski function \( f(z) = (z + 1/z)/2 \).
3. Let $f$ be a non-constant analytic function in some region. Can $|f|^2$ be harmonic in this region? (The answer must be justified: either you give an example of such function, or explain why it does not exist).
4. Evaluate with any method

\[ \int_{\gamma} \frac{\sin z}{z^2 - 1}, \]

where \( \gamma \) is the circle \( \{2e^{it} : 0 \leq t \leq 2\pi\} \).
5. Find a bounded harmonic function in the strip

\[ \{x + iy : -\infty < x < \infty, \ 0 < y < 1\} \]

which takes the value 1 on the positive ray \([0, +\infty)\) and 0 on the rest of the boundary of the strip. *Hint: use the exponential function to map the strip onto the upper half-plane*
6. Which of the following statements are true:

   a) If $u(z)$ is harmonic then $u(\overline{z})$ is harmonic.
   b) If $f(z)$ is analytic then $f(\overline{z})$ is analytic.
   c) If $f(z)$ is analytic then $\overline{f(\overline{z})}$ is analytic.
   d) If $u$ is harmonic then $u^2$ is harmonic.
   e) If $f$ is analytic then $f^2$ is analytic.

Here $\overline{x + iy} = x - iy$ is the complex conjugation.
No justification is necessary; each correct answer is worth 2 points.
7. Which of the following statements are true:

a) There is a branch of \((z^2 - 1)^{1/2}\) in the region \(\{ z : |z| > 1 \}\).

b) There is a branch of \((z^3 - 1)^{1/2}\) in the region \(\{ z : |z| > 1 \}\).

c) There is a branch of \(\log(z^2 - 1)\) in the region \(\{ z : |z| > 1 \}\).

d) There is a branch of \(\arccos z\) in the region \(\{ z : |z| < 1/100 \}\).

e) There is a branch of the inverse function to \(J(z) = (z + z^{-1})/2\) in the plane with deleted segment \([-1, 1]\).

No explanation necessary: every correct answer gives you 2 points.