Math 425/525 Midterm exam, Fall 2020.

NAME: Eremenko

1. Evaluate

$$2i\mathrm{Log}\,\frac{1-i}{1+i}.$$

Your answer should in the form $a+ib$ where a and b are real.

Solution.

$$2i\mathrm{Log}\,\frac{1-i}{1+i}=2i\mathrm{Log}\,(-i)=2i(0+i\mathrm{Arg}\,(-i))=2i(-\pi i/2)=\pi.$$

2. Describe and sketch the image of the sector

$$\{z : |z| < 1,\ \pi/2 < \operatorname{Arg} z < \pi\}$$

under the Joukowski function $f(z) = (z + 1/z)/2$.

Solution. We learned in class that f is one-to-one in the upper half-plane. Since the second quadrant belongs to the upper half-plane, it is one-to-one there, so it is sufficient to trace the image of the boundary. The boundary of the sector consists of three pieces: a) the interval $[0, i]$, b) the interval $[-1, 0]$ and c) the quarter of the circle. Their parametrizations:

a) $z = it,\ 0 \le t \le 1,$

b) $z = t,\ -1 \le t \le 0,$

c) $z = e^{it},\ \pi/2 \le t \le \pi.$

I choose parametrizations so that as t increases the region stays on the left. Plugging these to our function, we obtain parametrizations:

a) $i(t - 1/t)/2,\ 0 \le t \le 1$. This is the ray of the imaginary axis from $-\infty$ to 0.

b) $(t + 1/t)/2,\ -1 \le t \le 0$. This describes the part of real line from -1 to $-\infty$.

c) $(e^{it} + e^{-it})/2 = \cos t,\ \pi/2 \le t \le \pi$. This describes the interval from 0 to -1.

Putting all this together, we obtain the boundary of the third quadrant: $\{x+iy :\ x < 0, y < 0\}$. Since the region stays on the left of our parametrized curves, the image region is the third quadrant.

You could also refer to the fact mentioned in class that the image of the upper half of the unit disk is the lower half-plane. Then it is sufficient to find the image of a). It is the negative imaginary ray. So the region we are searching must be either 3-d or 4-th quadrant. To see which one, find the image of any one point which is not on the piece a).

3. Let f be a non-constant analytic function in some region. Can $|f|^2$ be harmonic in this region? (The answer must be justified: either you give an example of such function, or explain why it does not exist).

The answer is no. Justification. Suppose $f = u + iv$, where u and v are harmonic. Then $|f|^2 = u^2 + v^2$. To apply the Laplacian we compute partial derivatives:

$$(u^2)_{xx} = (2uu_x)_x = 2(u_x)^2 + 2uu_{xx},$$

Similarly,

$$(u^2)_{yy} = 2(u_y)^2 + 2uu_{yy}.$$

Since u is harmonic, $u_{xx} + u_{yy} = 0$, so

$$\Delta(u^2) = 2(u_x)^2 + 2(u_y)^2.$$

Similar computation gives

$$\Delta(v^2) = 2(v_x)^2 + 2(v_y)^2.$$

So

$$\Delta|f|^2 = 2(u_x^2 + u_y^2 + v_x^2 + v_y^2).$$

This can be 0 only if $u_x = u_y = v_x = v_y = 0$. This implies that the function f is constant.

Remark. Cauchy-Riemann conditions were not used! Only harmonicity of u and v.

4. Evaluate with any method

$$\int_\gamma \frac{\sin z}{z^2-1},$$

where γ is the circle $\{2e^{it} : 0 \le t \le 2\pi\}$.

Solution. By Cauchy's theorem, the integral is equal to the sum of two integrals: one over a little circle around 1 and another over a little circle around -1, both oriented counterclockwise.

Let us denote these two integrals by I_1 and I_{-1}. To I_1 we apply Cauchy integral formula with

$$f(z) = \frac{\sin z}{z+1}$$

and $a = 1$. We obtain $I_1 = 2\pi i f(1) = \pi i \sin 1$. To I_2 we apply Caychy formula with

$$f(z) = \frac{\sin z}{z-1}$$

and $a = -1$. We obtain $I_2 = -\pi i \sin(-1) = \pi i \sin 1$. So the answer is $I_1 + I_2 = 2\pi i \sin 1$.

Another method is to apply the partial fraction decomposition

$$\frac{1}{z^2-1} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right),$$

break the integralal into corresponding two parts, and evaluate each of them using Cauchy formula with $f(z) = \sin z$ and $a = \pm 1$.

5. Find a bounded harmonic function in the strip

$$\{x + iy : -\infty < x < \infty,\ 0 < y < 1\}$$

which takes the value 1 on the positive ray $[0, +\infty)$ and 0 on the rest of the boundary of the strip. *Hint: use the exponential function to map the strip onto the upper half-plane*

Solution. We learned in this class that e^z maps the strip

$$\{x + iy : 0 < y < \pi\}$$

onto the upper half-plane. The real line is mapped onto the positive ray and the line $y = \pi$ onto the negative ray. So the stip in the problem is mapped onto the upper half-plane by the function $e^{\pi z}$, and we reduce our problem to a Dirichlet problem in the upper half-plane. The given boundary function was 1 on the positive ray. Positive ray is mapped by $e^{\pi z}$ to the ray $[1, +\infty)$.

So we need to solve the Dirichlet problem for the upper half-plane, with the boundary data equal to 1 on the ray $[1, \infty)$ and zero on the rest of the boundary. Solution of such problems was explained in class. It uses the function Arg which is harmonic in the upper half-plane and takes values 0 on the positive ray and π on the negative ray. So the function

$$1 - \frac{1}{\pi}\mathrm{Arg}\,(z - 1)$$

has the correct boundary values.

Thus the solution of the original problem is

$$u(z) = 1 - \frac{1}{\pi}\mathrm{Arg}\ (e^{\pi z} - 1)\,.$$

Remark. Very few students solved this problem. I recommend to review Section 3.4, and solve all problems at the end of this section, especially problems 3-6.

6. Which of the following statements are true:

a) If $u(z)$ is harmonic then $u(\overline{z})$ is harmonic.

b) If $f(z)$ is analytic then $f(\overline{z})$ is analytic.

c) If $f(z)$ is analytic then $\overline{f(\overline{z})}$ is analytic.

d) If u is harmonic then u^2 is harmonic.

e) If f is analytic then f^2 is analytic.

Here $\overline{x + iy} = x - iy$ is the complex conjugation.
No justification is necessary; each correct answer is worth 2 points.

Answer: a), c, e) are true, the rest are false.

Explanation. For a), compute the Laplacian of $u(x, -y)$ and compare with the Laplacian of $u(x, y)$.

b) is false: for example, $f(z) = z$ is analytic but $\overline{z}$ is not. In fact when f is non-constant and analytic, $f(\overline{z})$ is never analytic. (This is another exercise).

c) is true: you can check that if $f = u + iv$, and u, v satisfy the Cauchy-Riemann conditions, then $\overline{f(\overline{z}} = u(x, -y) - iv(x, -y)$, where $z = x + iy$ also satisfies them.

d) false: see Problem 3 where Laplacian of u^2 was computed.

e) true: square of an analytic function is analytic (by the chain rule).

7. Which of the following statements are true:

a) There is a branch of $(z^2 - 1)^{1/2}$ in the region $\{z : |z| > 1\}$.

b) There is a branch of $(z^3 - 1)^{1/2}$ in the region $\{z : |z| > 1\}$.

c) There is a branch of $\log(z^2 - 1)$ in the region $\{z : |z| > 1\}$.

d) There is a branch of $\arccos z$ in the region $\{z : |z| < 1/100\}$.

e) There is a branch of the inverse function to $J(z) = (z + z^{-1})/2$ in the plane with deleted segment $[-1, 1]$.

No explanation necessary: every correct answer gives you 2 points.

Answers: the branches exist for: a), d) and e); for the rest they do not.

Explanation. a) and e) were discussed in class. For d), see the description of the branches of arccos in the textbook (they are refined in the region $\mathbf{C} \backslash ([-\infty, -1] \cup [1, +\infty))$, the plane with two rays removed.

For b), consider a circle of large radius. When a point moves on this circle and describes it once, the argument of this point increases by 2π, and argument of $z^3 - 1$ increases *approximately* by 6π, therefore the argument of $\sqrt{(z^3 - 1)}$ increases approximately by 3π. Since 3 is odd, the function does not return to the same value.

For c), the reason is the same, when we describe a large circle, the value of the function increases by approximately $4\pi i$, so the function does not return to its original value.