Math 425/525 Midterm exam, Fall 2022

1. a) How many complex solutions does this equation have?

$$(z+1)^5 = z^5.$$

b) Find all of them.

Solution. a) If you expand the LHS, z^5 cancels and you obtain a polynomial equation of degree 4. So there are 4 solutions, counting with multiplicity, and when we find them in b), we will see that all 4 are distinct.

b) Rewrite the equation as

$$\left(\frac{z+1}{z}\right)^5 = 1.$$

So

$$\frac{z_k+1}{z_k} = w_k = e^{2\pi ik/5}, \quad 0 \leq k \leq 4.$$

Solving this for z we obtain

$$z_k = \frac{1}{e^{2\pi ik/5} - 1}.$$

This makes no sense when $k = 0$, and for $1 \leq k \leq 4$ this formula gives four distinct solutions.

2. Find all solutions of the equation

$$\sin z = 2i$$

and make a picture of them.

Solution.

$$\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) = 2i.$$

denoting $e^{iz} = w$ we obtain

$$w - \frac{1}{w} = -4,$$

$$w^2 + 4w - 1 = 0;$$

and this quadratic equation has two solutions

$$w_1 = -2 + \sqrt{5} > 0 \quad \text{and} \quad w_2 = -2 - \sqrt{5} < 0.$$

Notice also that

$$|w_1 w_2| = 1, \tag{1}$$

which will be used to make an accurate picture. Now solving $e^{iz} = w_k,\ k = 1, 2$, we obtain 2 series of solutions

$$w_{1,k} = -i\text{Log}(-2 + \sqrt{5}) + 2\pi k, \quad k = 0, \pm 1, \pm 2, ...$$

and, since $\arg(-2 - \sqrt{5}) = \pi$,

$$w_{2,k} = -i\text{Log}(2 + \sqrt{5}) + \pi + 2\pi k, \quad k = 0, \pm 1, \pm 2, \ldots.$$

To make the accurate picture, we notice that

$$0 < -2 + \sqrt{5} < 1$$

so its logarithm is negative, while

$$2 + \sqrt{5} = \frac{1}{-2 + \sqrt{5}},$$

so $\text{Log}(1 + \sqrt{5})$ is $-\text{Log}(-2 + \sqrt{5})$, that is positive and of the same absolute value.

3. Does there exist a real non-constant harmonic function u in some region, such that u^2 is also harmonic?

If the answer is positive, give an example, if negative explain why.

Solution. The answer is negative. Differentiate:

$$(u^2)_x = 2uu_x, \quad (u^2)_{xx} = 2u_x^2 + 2uu_{xx}.$$

Similarly

$$(u^2)_{yy} = 2u_y^2 + 2uu_{yy}.$$

Adding the results and using $u_{xx} + u_{yy} = 0$, we obtain that

$$\Delta(u^2) = 2(u_x^2 + u_y^2) = 0,$$

which implies $u_x = u_y = 0$, so u is constant.

4. Does there exist an analytic function in the whole plane whose real part is

$$u(x, y) = (\sin x)(e^y + e^{-y}).$$

If the answer is positive, find *all* such functions, if negative, explain why.

Solution. Yes, they exist since $u_{xx} + u_{yy} = 0$, and the whole plane is simply connected. Let this analytic function be $f = u + iv$.

Using the first Cauchy Riemann equation, we have:

$$u_x = \cos x(e^y + e^{-y}) = v_y,$$

Integrating with respect to y, we obtain

$$v = \cos x(e^y - e^{-y}) + C(x).$$

Differentiate with respect to x and write that this is equal to $-u_y$ (the second Cauchy–Riemann equation):

$$v_x = -\sin x(e^y - e^{-y}) + C'(x) = -u_y = -\sin x(e^y - e^{-y}),$$

so $C(x) = C_0$, a real constant, and the general form of our analytic function is

$$f(x + iy) = \sin x(e^y + e^{-y}) + i \cos x(e^y - e^{-y}) + iC_0.$$

5. Find a bounded harmonic function u in the first quadrant

$$\{z = x + iy : x > 0,\ y > 0\}$$

which takes the boundary values 1 on the ray $[1, +\infty)$ and 0 on the rest of the boundary.

Solution. Since we know how to solve the Dirichlet problem for the upper half-plane, let us find an analytic function which maps our region (the first quadrant) onto the upper half-plane in a one-to-one manner. This function is $f(z) = z^2$. The image of the ray $[1, +\infty)$ is the same ray $[1, \infty)$. So we need as bounded function harmonic in the upper half-plane which takes the value 1 on the ray $[1, +\infty)$ and zero on the rest of the boundary. This function is

$$1 - \frac{1}{\pi}\text{Arg}(z - 1).$$

Putting everything together, we obtain the answer

$$u(z) = 1 - \frac{1}{\pi}\text{Arg}(z^2 - 1).$$

6. a) For every integer m (positive or negative, or zero), evaluate the integral

$$\int_\gamma \overline{z}^m \, dz,$$

where γ is the circle $|z| = 2$, oriented counterclockwise.

b) When $m > 0$, are these integrals path-independent in the whole plane? Hint: derivative of an analytic function is also analytic, which can be verifyed by using CR equations.

Solutiona. a) Parametrize the curve as $z(t) = 2e^{it}$, $0 \leq t \leq 2\pi$. Then $dz = 2ie^{it}$ and our integral is equal (by definition!)

$$2^{m+1} i \int_0^{2\pi} e^{i(1-m)t} dt = \begin{cases} 0, & m \neq 1, \\ 8i\pi, & m = 1. \end{cases}$$

b) No. If such an integral were path independent, then it would have a primitive, so we would have an analytic function F such that $F'(z) = \overline{z}^m$. But the derivative of an analytic function is always analytic (there was a hint about this in the exam). This was a theorem proved in class, but in a lecture which is not covered by the exam. For this reason I gave a hint. to verify the hint, write $F = u + iv$, $F' = u_x + iv_x$, and this satisfies the Cauchy-Riemann condition: just differentiate both Cauchy–Riemann equations for u, v once with respect to x. This proves the hint.

To see that $\overline{z}^m$ is not analytic, restrict your function $\overline{z}^m$ to a small angular sector, say $\{z : |\text{Arg} z| < \pi/m\}$ then it is one-to-one in this sector and maps it to the RHP. If this were an analytic function, say ϕ, then $\overline{z} = (\phi(z))^{1/m})$, where $z^{1/m}$ is an analytic branch of the inverse to z^m in the RHP, and this is a contradiction since composition of two analytic functions is analytic, while $\overline{z}$ is not analytic, as we know.

7. Find the image of the segment $[0, (1+i)/2]$ under the Joukowski function

$$J(z) = \frac{1}{2}\left(z + \frac{1}{z}\right).$$

Make a picture of this image.

Hint: is is a part of some hyperbola. Write the equation of this hyperbola, determine which part of it is our curve, and make a picture.

Solution. Parametrize the segment as $z(t) = (1+i)t/2 : 0 \leq t \leq 1$. Then the image is parametrized by

$$\frac{(1+i)t}{4} + \frac{1}{(1+i)t} = \frac{(1+i)t}{4} + \frac{(1-i)}{2t}$$

or

$$x(t) = \frac{t}{4} + \frac{1}{2t}, \qquad y(t) = \frac{t}{4} - \frac{1}{2t}.$$

We have to eliminate t. Since there was a hint that this is a hyperbola, try to compute $x^2 - y^2$. We obtain

$$x^2 - y^2 = \frac{1}{2}.$$

So our curve is indeed a part of this hyperbola, but not the whole hyperbola. Our curve certainly consists of one piece, since it is parametrized by a continuous function, and to find it exactly we find the images of the extremities of our arc. They correspond to $t = 0$ and $t = 1$. To $t = 0$ corresponds the point ∞ on our hyperbola, and to the point $t = 1$ corresponds the point $x + iy = (3/4) - i/4)$. So the image is the unbounded curve on the right piece of the hyperbola, from this point to infinity, below the real axis. Below because $y(t) = t/4 - 1/(2t) < 0$ for $t \in (0, 1)$.