Practice exam solutions

1. $|-2i(3+i)(2+4i)(1+i)| = 2\sqrt{(9+1)(4+16)(1+1)} = 2\sqrt{400} = 40.$

2. The equality says that sum of the distances from z to a and $-a$ is $2R$. This describes an ellips, whose foci are at a and $-a$. The ellipse is symmetric with respect to the origin, so the largest valkue of $|z|$ is its large semiaxis, that is $|z| = R$, and the smallest is $\sqrt{R^2 - |a|^2}$.

3. Joukowski function is one-to one in the upper half-plane, so it is enough to look at the image of the boundary, which consists of the positive ray and the positive-imaginary ray. The positive ray is mapped onto the ray from the point 1 to the right, passed twice, and the imaginary ray onto the whole imaginary axis. So the image is the right halph-plane, slit along the horizontal ray to the right from the point 1.

4. Evaluating the area.

$$A = \int\int_{|z-1|\leq 1} |f'(z)|^2 da = 4\int\int_{|z-1|\leq 1} |z|^2 da.$$

In polar coordinates the disc is described by $0 \leq r \leq 2\cos\phi$ for $-\pi/2 \leq \phi \leq \pi/2$, so

$$A = 4\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\phi} r^3\, dr\, d\phi = \int_{-\pi/2}^{\pi/2} 2^4 \cos^4\phi\, d\phi.$$

If you are not permitted to use tables of integrals, the simplest way to do this is to use complex numbers:

$$2^4\cos^4\phi = (e^{i\phi}+e^{-i\phi})^4 = e^{4i\phi}+4e^{2i\phi}+6+4e^{-2i\phi}+e^{-4i\phi} = 2\cos 4\phi+8\cos 2\phi+6.$$

So $A = 6\pi$.

5. It cannot. Use the Laplace equation and the Cauchy-Riemann equations to conclude that $f = \text{const}$.

6. The first equation is similar to $\cos z = 2$, which was solved in class. Solving the second.

$$\cos z = \frac{w + w^{-1}}{2}, \quad \text{where} \quad w = e^{iz}.$$

We have

$$\frac{w + w^{-1}}{2} = -i, \quad w^2 + 2iw + 1 = 0,$$

$$w = -i \pm \sqrt{-2} = i(-1 \pm \sqrt{2}),$$

where the last square root is the arithmetic one. Now $e^{iz} = i(1 \pm \sqrt{2})$, so

$$z = -i \log(i(-1 \pm \sqrt{2}) = \pm(i\mathrm{Log}(1 + \sqrt{2})\pi/2) + 2\pi k,$$

where k is any integer. Draw a picture!

7. First one:

$$\int_{|z|=1} \Re z \, dz = \int_0^{2\pi} (\cos t)(ie^{it})dt = (i/2) \int_0^{2\pi} (e^{2it} + 1)dt = \pi i.$$

Second one:

$$\int_0^{2\pi} |e^{it} - 1|^2 (ie^{it})dt = i \int_0^{2\pi} (2 - 2\cos t)e^{it}dt = -2\pi i.$$

9. First is true, others are not.