Linear independence of exponentials on the real line

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August 25, 2013

The following question was asked on Math Overflow. Let (λ_n) be a sequence of complex numbers tending to infinity. Exponential functions $e^{\lambda_n z}$ are called **R**- (linearly) dependent if there are complex numbers a_n, not all equal to zero, such that the series

$$f(z) = \sum_n a_n e^{\lambda_n z} \tag{1}$$

converges to zero uniformly on compact subsets of the real line $\mathbf{R}$.

The question is under what conditions on λ_n the exponentials are linearly independent.

Let us recall some known results. Let

$$D = \limsup_{r \to \infty} \frac{\#\{n : |\lambda_n| \leq r\}}{r} \tag{2}$$

be the upper density. If $D < \infty$ and the series (1) converges uniformly on compact subsets of the complex plane $\mathbf{C}$ then $a_n = 0$ for all n, [2, 3, 1]. On the other hand, there are sequences of infinite density, in fact with the quotient in the RHS of (2) growing arbitrarily slowly, such that some series (1) with non-zero coefficients converges to zero uniformly on compact subsets of $\mathbf{C}$, see [2, 3, 1].

With a different notion of linear independence, stated in the beginning, one can obtain very complete results. If one of the exponentials does not belong to the closure of the linear span of the rest, then $i\Lambda$ is a subset of the zero set of the Fourier transform of a measure with bounded support on the real line. And conversely, if $i\Lambda$ is the zero set of such a Fourier transform than no exponential of the set belongs to the closure of the linear span of the rest. These results belong to L. Schwartz [5]. Zeros of Fourier transforms of measures with bounded support have finite upper density. The requirement that one of the exponentials is in the closure of the linear span of the others is of course much weaker than the requirement that the series (1) converges to zero.

The proof these results of Schwartz is simple, so we include it. Let C be the space of all continuous functions $\mathbf{R} \to \mathbf{C}$ with topology of uniform convergence on compact subsets. The dual space C' consists of Borel measures with compact support. Suppose that $e^{\lambda_0 z} \notin S$, where S is the closure of the span of $\{e^{\lambda_n z} : n \geq 1\}$. Then $S \neq C$, and thus there exists a measure $\mu \in C'$ such that

$$\int g(x) d\mu = 0 \quad \text{for all} \quad f \in S.$$

Applying this to our exponentials, we obtain

$$\int e^{\lambda_n x} d\mu = 0, \quad n \geq 1,$$

that is $M(i\lambda_n) = 0$, where

$$M(\lambda) = \int e^{-i\lambda x} d\mu$$

is the Fourier transform of μ.

Now consider the function $M(i\lambda_1 - i\lambda_0 + \lambda)M(\lambda)$. This is also a Fourier transform of some measure with compact support, and its zero set contains all $i\lambda_n,\ n \geq 0$.

To prove the converse statement, let Φ be the Fourier transform of some measure μ with compact support in $\mathbf{R}$. Then Φ is an entire function of exponential type, bounded on the real line. Let $i\lambda_j$ be the zeros of Φ. For each n, we define the entire function $\Phi_n(\lambda) = \Phi(\lambda)/(\lambda - i\lambda_n)$. This function belongs to $L^2(\mathbf{R})$ because Φ is bounded on $\mathbf{R}$. So by the Wiener–Paley theorem, Φ_n is the Fourier transform of some measure $\mu_n \in C'$. For this measure we have

$$\int e^{\lambda_k x} d\mu_n = \begin{cases} 0, & k \neq n, \\ \Phi'(i\lambda_n), & k = n. \end{cases}$$

So $e^{\lambda_n x}$ cannot be in the closed span of the rest.

In this note we prove the following.

Theorem. *Let Λ be a sequence of finite density. If the series (1) is absolutely and uniformly convergent to zero on compact subsets of the real line, then all $a_n = 0$.*

It is not clear whether one can relax the condition of absolute convergence in this theorem. On the other hand, this condition seems natural. Indeed, linear independence of finitely many vectors is a property of an *unordered* set of vectors, so it is natural that an extension of this property to an infinite set of vectors be a property of the *set* not a *sequence* of vectors.

Proof. We begin by partitioning Λ into three subsets

$$\Lambda = \Lambda_0 \cup \Lambda_1 \cup \Lambda_2,$$

where

$$\Lambda_0 = \{\lambda \in \Lambda : |\mathrm{Re}\,\lambda| \geq |\mathrm{Im}\,\lambda|\} \cup \{\lambda \in \Lambda : |\lambda| \leq 1\},$$

Λ_1 and Λ_2 are the rest of Λ in the upper and lower half-planes, respectively. Now we partition our series correspondingly:

$$f = f_0 + f_1 + f_2.$$

Lemma 1. *The series f_0 converges uniformly on compact subsets of* $\mathbf{C}$.

Proof. If $\mathrm{Re}\,\lambda_n > 0$, we have

$$|a_n||e^{\lambda_n z}| \leq |a_n|e^{|\lambda_n||z|} \leq |a_n|e^{\sqrt{2}|\mathrm{Re}\,\lambda_n||z|} = |a_n e^{\lambda_n(\sqrt{2}|z|+1)}||e^{-\lambda_n}|.$$

Then $|a_n e^{\lambda_n(\sqrt{2}|z|+1)}| \to 0$ because the series f_0 converges on the real line, and in addition, $|e^{-\lambda_n}| = O(e^{-n/K})$, for some $K > 0$, because the upper density is finite. If $\mathrm{Re}\,\lambda_n < 0$, we apply similar argument and obtain

$$|a_n||e^{\lambda_n z}| \leq |a_n e^{\lambda_n(-\sqrt{2}|z|-1)}||e^{\lambda_n}|.$$

There are only finitely many terms with $\mathrm{Re}\,\lambda_n = 0$. Thus the series converges uniformly (and absolutely) on every compact subset of the complex plane.

Lemma 2. *The series f_1 converges uniformly on compact subsets of the lower half-plane to an analytic function F_1 in the lower half-plane, continuous in the closed lower half-plane. Similarly, f_2 converges in the lower half-plane to an analytic function F_2 continuous in the closed lower half-plane.*

Proof. Fix an arbitrary point $x_0 \in \mathbf{R}$. First we notice that the series f_1 is uniformly convergent in the closed sector

$$T(x_0) = \{z : |\arg(z - x_0) + \pi/2| \leq \pi/8\} \cup \{x_0\}.$$

Indeed, for z in this sector we have

$$|a_n e^{\lambda_n z}| \leq |a_n e^{\lambda_n x_0}|.$$

This was the only place where the absolute convergence was used. It follows that f_1 converges uniformly on compact subsets in the lower half-plane to an analytic function F_1. This function has angular limits everywhere on the real line, and these angular limits make a continuous function on the real line (because f_1 is uniformly convergent on compact subsets of the real line). Then it follows from the Poisson representation that F_1 is continuous in the closed lower half-plane. The proof for f_2 and F_2 is similar.

Thus we have three functions F_0 (which is the limit of the series f_0), F_1 and F_2, where F_0 is entire, F_1 is analytic in the lower half-plane, continuous in the closed lower half-plane, and F_2 analytic in the upper half-plane and copntinuous in the closed upper half-plane. Moreover, on the real line, where all three functions are defined, we have $F_0 + F_1 + F_2 = 0$. It follows from the removable singularity theorem for continuous functions that in fact all three functions are entire, and the relation

$$F_0 + F_1 + F_2 = 0 \tag{3}$$

holds in $\mathbf{C}$.

Now we fix some n, and suppose that $\lambda_n \in \Lambda_1$. Let Φ_n be an entire function of exponential type whose zeros are $\{\lambda_j \in \Lambda : j \neq n\}$ and $\Phi_n(\lambda_n) = 1$. Let K be the conjugate indicator diagram of Φ_n, and γ a positively oriented circle which encloses K. It is easy to see that K and γ can be chosen independently of n, but this is irrelevant for our argument. Let ϕ_n be the Laplace transform of Φ_n. This is an analytic function in $\mathbf{C}\backslash K$, and $\phi(\infty) = 0$. Then Φ_n is the Borel transform (see [2, 3, 4]) of ϕ_n:

$$\Phi_n(\lambda) = \frac{1}{2\pi i} \int_\gamma \phi_n(z) e^{\lambda z} dz.$$

Consider the integral

$$g_n(w) = 2\pi i \int_\gamma F_1(z + w) \phi_n(z) dz. \tag{4}$$

Evidently, this is an entire function. Let C_1 be a real constant such that $\gamma + iC_1$ is in the lower half-plane. Then for $\operatorname{Im} w < C_1$, the function F_1 in the integral is the sum of the uniformly convergent series f_1. Substituting this series and integrating term by term, we obtain

$$g_n(w) = \sum_{\lambda_k \in \Lambda_1} a_k e^{\lambda_k w} \Phi_n(\lambda_k) = a_n e^{\lambda_n w}. \tag{5}$$

Now let C_2 be a real constant such that $\gamma + iC_2$ is in the upper half-plane. Then for $\operatorname{Im} w > C_1$, the function $F_1 = -F_0 - F_2$ is the sum of uniformly convergent series $-f_0 - f_2$. Substituting this series for F_1 into (4) and integrating term-by-term, we obtain

$$g_n(w) = \sum_{k \in \Lambda_0 \cup \Lambda_2} a_k e^{\lambda_k w} \Phi_n(\lambda_k) = 0.$$

Comparing this with (5), we obtain that that $a_n = 0$. The same argument can be applied to $\lambda_n \in \Lambda_2$. Thus we obtain that $f_1 = f_2 = 0$ (as formal series), and $f = f_0$. But f_0 converges to zero in $\mathbf{C}$ and Leontiev's theorem implies that f_0 is also zero.

References

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