Examples of Taylor and Laurent series expansions

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1. The first and most important example is the geometric progression formula

$$\frac{1}{1-z} = \sum_0^\infty z^n = 1 + z + z^2 + z^3 + \ldots. \tag{1}$$

This can be differentiated any number of times:

$$\frac{1}{(1-z)^2} = \sum_0^\infty (n+1) z^n = 1 + 2z + 3z^2 + 4z^3 + \ldots. \tag{2}$$

$$\frac{1}{(1-z)^3} = \frac{1}{2} \sum_0^\infty (n+1)(n+2) z^n = 1 + 3z + 6z^2 + 10z^3 + \ldots. \tag{3}$$

And so on. These examples permit to expand any rational function at any point: first do partial fraction decomposition, then use these formulas.

Example 1. Expand into a Laurent series in $0 < |z| < 1$:

$$f(z) = \frac{1}{z(z^2-1)} \tag{4}$$

Solution. We have

$$\frac{1}{z^2-1} = \frac{1}{2} \left(\frac{1}{z-1} - \frac{1}{z+1} \right). \tag{5}$$

The first summand gives

$$\frac{1}{z-1} = -\sum_{n=0}^\infty z^n = -1 - z - z^2 - z^3 - \ldots.$$

by formula (1). Same formula gives

$$\frac{1}{z+1} = \sum_{n=0}^{\infty}(-1)^n z^n = 1 - z + z^2 - z^3 + \ldots.$$

Subtracting second from the first, dividing by 2 and by z, we obtain

$$f(z) = \frac{1}{z(z^2-1)} = -\sum_{n=0}^{\infty} z^{2n-1} = -z^{-1} - z - z^3 - \ldots.$$

Example 2. Expand the same function in the region $0 < |z-1| < 1$ First we rewrite it in terms of $w = z-1$: we have $z = w+1$

$$f(z) = \frac{1}{(w+1)((w+1)^2-1)} = \frac{1}{w(w+1)(w+2)}.$$

Then decompose into partial fraction:

$$f(z) = \frac{1}{2w} - \frac{1}{w+1} + \frac{1}{2(w+2)}.$$

The first term is already expanded in powers of w. For the second term we have by formula (1)

$$-\frac{1}{w+1} = -\sum_{n=0}^{\infty}(-1)^n w^n = \sum_{n=0}^{\infty}(-1)^{n-1}w^n.$$

For the third term, by the same formula (1):

$$\frac{1}{2(w+2)} = \frac{1}{4(1+w/2)} = \frac{1}{4}\sum_{n=0}^{\infty} 2^{-n}w^n.$$

combining all together, we obtain that

$$f(z) = \frac{1}{2w} + \sum_{n=0}^{\infty}((-1)^{n-1} + 2^{-n-2})w^n = \frac{1}{2(z-1)} + \sum_{n=0}^{\infty}((-1)^{n-1} + 2^{-n-2})(z-1)^n.$$

Taylor formula is also a powerful method of expansion. Once you represented a rational function as a sum of partial fractions, differentiation becomes easy, you can compute all derivatives, and obtain the expansion.

2. Laurent series at ∞.

Example 3. Expand the same function f is Example 1 into a Laurent series in the ring $1 < |z| < \infty$.

Solution.
$$\frac{1}{z(z^2-1)} = -z^{-3}\frac{1}{1-1/z^2} = z^{-3}\sum_{n=0}^{\infty} z^{-2n}.$$
Notice that we always take our of parentheses in the denominater the term of the bigger absolute value so tat the resulting geometric series converges.

The following expansions are recommended to remember

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \ldots.$$

$$\sin z = \sum_{m=0}^{\infty} (-1)^m \frac{z^{2m}}{(2m)!} = z - \frac{z^3}{6} + \frac{z^5}{120} - \ldots.$$
$$\cos z = \sum_{m=0}^{\infty} (-1)^m \frac{z^{2m+1}}{(2m+1)!} = 1 - \frac{z^2}{2} + \frac{z^4}{24} - \ldots.$$
$$\mathrm{Log}\,(1+z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n} = z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots.$$

Laurent series expansions can be added. They can be differentiated and integrated term-by-term. Meromorphic Laurent series can be also multilied: the formulas for the n-th coefficient of product is a finite sum in terms of coefficients of the multiples. We have

$$\left(\sum_{n=0}^{\infty} a_n z^n\right)\left(\sum_{n=0}^{\infty} b_n z^n\right) = \sum_{n=0}^{\infty} c_n z^n,$$

where
$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$
Meromorphic Laurent series can be also divided. The usual tool is the same geometric progression formula.

Example 4. Find few first terms of the Laurent expansion in powers of z of the function
$$f(z) = \frac{1}{\sin z}.$$

We substitute the series for sin and transform so that formula (1) can be used:

$$f(z) = \frac{1}{z - z^3/6 + z^5/120 - z^7/5040 + \ldots} = \frac{1}{z\,(1 - (z^2/6 - z^4/120 + z^6/5040 - \ldots))}$$

$$= z^{-1}\left(1 + (z^2/6 - z^4/120 + z^6/5040 - \ldots) + (z^2/6 - z^4/120 + \ldots)^2 + (z^2/6 + \ldots)^3 \ldots\right)$$

$$= z^{-1} + z/6 + z^3(-1/120 + 1/36) + z^5(1/5040 - 2/(6 \cdot 120) + 1/6^3) + \ldots$$

$$= z^{-1} + z/6 + 7x^3/360 + 31x^5/15120 + \ldots.$$