Linear dependence of exponentials

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Abstract

Any finite set of distinct exponentials is linearly independent. People ask me whether infinitely many exponentials $\exp(\lambda_k z)$ are linearly independent. The answer is "well known" but the only reference I have is in Russian. People ask so frequently that I decided to post the answer on the web. A complete study of these questions can be found in the books of the Russian mathematician A. Leontiev.

Dear Steven and Sherman,

I recently saw the problem on "linear independence of exponentials" posted on Steven's web page, with my comments, and decided to send more specific comments. Here they are:

1. Let (λ_n) be a given sequence of exponents. Consider an entire function $L(\lambda)$ which has simple zeros exactly at λ_n. For example, we can take L to be the canonical product with these zeros). By the Residue Theorem

$$\sum_{k:|\lambda_k|<r} \frac{e^{\lambda_k z}}{L'(\lambda_k)} = \frac{1}{2\pi i} \int_{|\lambda|=r} \frac{e^{\lambda z}}{L(\lambda)} d\lambda. \tag{1}$$

2. Now it is easy to construct a sequence (λ_n) such that:
a) The series

$$\sum_{k=1}^{\infty} \frac{e^{\lambda_k z}}{L'(\lambda_k)}$$

is absolutely convergent, and
b) the integral in the right hand side of (1) tends to zero as $r \to \infty$, uniformly when z is restricted to any compact set.

The simplest example can be obtained by taking λ_n at the lattice points $\{m+in : m,n,\in \mathbf{Z}\}$. Then L is a Weierstrass sigma-function, and

$$\log|L'(\lambda_k)| \sim c|\lambda_k|^2$$

with some positive constant c. Furthermore,

$$\log|L(\lambda)| \sim c|\lambda|^2,$$

when $\lambda \to \infty$ avoiding exponentially small neighborhoods of the points λ_k. Thus properties a) and b) above are satisfied.

3. The main positive results are the following.

A. Suppose that $k = O(\lambda_k)$ and a series

$$\sum_{k=1}^{\infty} a_k \exp(\lambda_k z) \equiv 0 \tag{2}$$

converges uniformly and absolutely in the whole plane. Then $a_k = 0$. This follows from Theorem 3.1.4 on p. 201 of Leontiev's book: Series of Exponentials, Moscow, 1976.

B. Suppose that all λ_k are real, and the series (2) converges in the whole complex plane to 0. Then $a_k = 0$. This can be found in many books, for example, in S. Mandelbrojt, Séries de Dirichlet. Paris, Gauthier-Villars, 1969, Th. I.3.1.

4. Here is a brief proof of Leontyev's theorem. Let λ_k and L be the same as before, and put

$$L_k(\lambda) = \frac{L(\lambda)}{\lambda - \lambda_k},$$

so that

$$L_k(\lambda_n) = \begin{cases} 0, & n \neq k, \\ L'(\lambda_k), & n = k \end{cases} \tag{3}$$

Let f_k be the Laplace (Borel) transform of L_k and K the conjugate diagram of L. (See, for example, B. Levin, Lectures on Entire Functions, or any other book on entire functions). Take any closed curve γ going once around K, multiply (2) by f_k and integrate along this curve. A uniformly and absolutely convergent series can be integrated term-by-term, so

$$0 = \sum_{j=1}^{\infty} a_j \int_\gamma f_k(w) e^{\lambda_j w} dw = a_k L_k(\lambda_k),$$

so $a_k=0$ because $L_k(\lambda_k)=L'(\lambda_k)\neq 0$, while $L_k(\lambda_j)=0$ for $j\neq k$.

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