A version of Fabry's theorem for power series with regularly varying coefficients

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Abstract

For real power series whose non-zero coefficients satisfy $|a_m|^{1/m} \to 1$, we prove a stronger version of Fabry's theorem relating the frequency of sign changes in the coefficients and analytic continuation of the sum of the power series. AMS Subj. Class.: 30B10, 30B40.

For a set Λ of non-negative integers, we consider the counting function

$$n(x, \Lambda) = \#\Lambda \cap [0, x].$$

We say that Λ is *measurable* if the limit

$$\lim_{x \to +\infty} n(x, \Lambda)/x$$

exists, and call this limit the *density* of Λ.

Let $S = \{a_m\}$ be a sequence of real numbers. We say that *a sign change* occurs at the place m if there exists $k < m$ such that $a_m a_k < 0$ while $a_j = 0$ for $k < j < m$.

Theorem A. *Let Δ be a number in $[0, 1]$. The following two properties of a set Λ of positive integers are equivalent:*

(i) Every power series

$$f(z) = \sum_{m=0}^{\infty} a_m z^m \tag{1}$$

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of radius of convergence 1*, with real coefficients and such that the changes of sign of* $\{a_m\}$ *occur only for* $m \in \Lambda$*, has a singularity on the arc*

$$I_\Delta = \{e^{i\theta} : |\theta| \leq \pi\Delta\},$$

and

(ii) For every $\Delta' > \Delta$ *there exists a measurable set* $\Lambda' \subset \mathbf{N}$ *of density* Δ' *such that* $\Lambda \subset \Lambda'$*.*

Implication (ii) $\longrightarrow$ (i) is a consequence of Fabry's General Theorem [6, 3], as restated by Pólya. For the implication (i) $\longrightarrow$ (ii) see [9]. Fabry's General Theorem takes into account not only the sign changes of coefficients but also the absolute values of coefficients. It has a rather complicated statement and the sufficient condition of the existence of a singularity given by this theorem is not the best possible. The best possible condition in Fabry's General Theorem is unknown, see, for example the discussion in [4].

Alan Sokal (private communication) asked what happens if we assume that the power series (1) satisfies the additional regularity condition:

$$\lim_{m \in P, m \to \infty} |a_m|^{1/m} = 1, \tag{2}$$

where $P = \{m : a_m \neq 0\}$. This condition holds for most interesting generating functions. The answer is somewhat surprising:

Theorem 1. *Let* Δ *be a number in* $[0, 1]$*. The following two properties of a set* Λ *of positive integers are equivalent:*

a) Every power series (1) *satisfying* (2)*, with real coefficients and such that the changes of sign of the coefficients* a_m *occur only for* $m \in \Lambda$*, has a singularity on the arc* I_Δ*, and*

b) All measurable subsets $\Lambda' \subset \Lambda$ *have densities at most* Δ*.*

We recall that the *minimum density*

$$\underline{D}_2(\Lambda) = \lim_{r \to 0+} \liminf_{x \to +\infty} \frac{n((r+1)x, \Lambda) - n(x, \Lambda)}{rx}$$

can be alternatively defined as the sup of the limits

$$\lim_{x \to \infty} n(x, \Lambda')/x \tag{3}$$

over all measurable sets $\Lambda' \subset \Lambda$.

Similarly the *maximum density* of Λ is

$$\overline{D}_2(\Lambda) = \lim_{r\to 0+} \limsup_{x\to\infty} \frac{n((r+1)x,\Lambda) - n(x,\Lambda)}{rx},$$

and it equals to the inf of the limits (3) over all measurable sequences of non-negative integers Λ' containing Λ.

For all these properties of minimum and maximum densities see [12].

Thus condition (ii) is equivalent to $\overline{D}_2(\Lambda) \leq \Delta$ while condition b) is equivalent to $\underline{D}_2(\Lambda) \leq \Delta$.

Here is a gap version of Theorem 1:

Theorem 2. *The following two properties of a set Λ of positive integers are equivalent:*

A. Every power series

$$\sum_{m\in\Lambda} a_m z^m \tag{4}$$

satisfying (2) *has a singularity on I_Δ,*

A′. Every power series (4) *satisfying* (2) *has a singularity on every closed arc of length $2\pi\Delta$ of the unit circle, and*

B. $\underline{D}_2(\Lambda) \leq \Delta$.

The equivalence between A and A' is immediate as all assumptions of the statement A are invariant with respect to the change of the variable $z \mapsto \lambda z,\ |\lambda| = 1$.

Proof of Theorem 1. b) $\longrightarrow$ a).

Proving this by contradiction, we assume that $\underline{D}_2(\Lambda) \leq \Delta$, and there exists a function f of the form (1) with the property (2) which has an analytic continuation to I_Δ, and such that the sign changes occur only for $m \in \Lambda$.

Without loss of generality we assume that $a_0 = 1$, and $\Delta < 1$.

Lemma 1. *For a function f as in* (1) *to have an immediate analytic continuation from the unit disc to the arc I_Δ it is necessary and sufficient that there exists an entire function F of exponential type with the properties*

$$a_m = (-1)^m F(m), \quad \textit{for all} \quad m \geq 0, \tag{5}$$

and

$$\limsup_{t\to\infty} \frac{\log|F(te^{i\theta})|}{t} \le \pi b|\sin\theta|, \quad |\theta| < \alpha, \tag{6}$$

with some $b < 1 - \Delta$ and some $\alpha \in (0, \pi)$.

This result can be found in [1], see also [2, 4].

Consider the sequence of subharmonic functions

$$u_m(z) = \frac{1}{m}\log|F(mz)|, \quad m = 1, 2, 3, \ldots. \tag{7}$$

This sequence is uniformly bounded from above on every compact subset of the plane, because F is of exponential type. Moreover, $u_m(0) = 0$ because of our assumption that $a_0 = F(0) = 1$. Compactness Principle [8, Th. 4.1.9] implies that from every sequence of integers m one can choose a subsequence such that the limit $u = \lim u_m$ exists. This limit is a subharmonic function in the plane that satisfies in view of (6)

$$u(re^{i\theta}) \le \pi br|\sin\theta|, \quad |\theta| < \alpha, \tag{8}$$

with some b satisfying

$$0 < b < 1 - \Delta.$$

We use the following result of Pólya [11, footnote 18, p. 703]:

Lemma 2. *Let f be a power series* (1) *of radius of convergence* 1*. Let $\{a_{m_k}\}$ be a subsequence of coefficients with the property*

$$\lim_{k\to\infty} |a_{m_k}|^{1/m_k} = 1,$$

and assume that for some $r > 0$ the number of non-zero coefficients a_j on the interval $m_k \le j \le (1 + r)m_k$ is $o(m_k r)$ as $k \to \infty$. Then f has no analytic continuation to any point of the unit circle.

Lemma 2 also follows from the results of [1] or [4].

Now we show that (2) implies the following:

Lemma 3. *Every limit function has the property $u(x) = 0$ for $x \ge 0$.*

Proof of Lemma 3. Let $U = \{x : x \ge 0, u(x) < 0\}$. This set is open because u is upper semi-continuous. Take any closed interval $J = [c, d] \subset U$.

Then $u(x) \leq -\epsilon$, $x \in J$, with some $\epsilon > 0$. Let $\{m_k\}$ be the sequence of integers such that $u_{m_k} \to u$. Then from the definition of u_m we see that

$$\log |F(m_k x)| \leq -m_k \epsilon/2 \quad \text{for} \quad x \in J$$

and for all large k. Together with (5) and (2) this implies that $a_j = 0$ for all $j \in m_k J$. Let $a_{m'_k}$ be the last non-zero coefficient before cm_k. Applying Lemma 2 to the sequence $\{m'_k\}$ we conclude that f has no analytic continuation from the unit disc. This is a contradiction which proves Lemma 3. □

Now we use the following general fact:

Grishin's Lemma. *Let $u \leq v$ be two subharmonic functions, and μ and ν their respective Riesz measures. Let E be a Borel set such that $u(z) = v(z) > -\infty$ for $z \in E$. Then the restrictions of the Riesz measures on E satisfy*

$$\mu|_E \leq \nu|_E.$$

The references are [13, 7, 5].

In view of Lemma 2, we can apply Grishin's Lemma to u and $v(z) = \pi b|\text{Im } z|$ and $E = [0, \infty) \subset \mathbf{R}$. We obtain that the Riesz measure $d\mu$ of any limit function u of the sequence $\{u_k\}$ satisfies

$$d\mu|_{[0,\infty)} \leq b\ dx. \tag{9}$$

Now we go back to our coefficients and function F. By our assumption, the sign changes occur on a sequence Λ whose minimum density is at most Δ. Choose a number a such that $b < a < 1 - \Delta$. By the first definition of the minimum density, there exist $r > 0$ and a sequence $x_k \to \infty$ such that

$$n((1+r)x_k, \Lambda) - n(x_k, \Lambda) \leq (1-a)rx_k.$$

Lemma 4. *Let $(a_0, a_1, \ldots, a_N)$ be a sequence of real numbers, and f a real analytic function on the closed interval $[0, N]$, such that $f(n) = (-1)^n a_n$. Then the number of zeros of f on $[0, N]$, counting multiplicities, is at least N minus the number of sign changes of the sequence $\{a_n\}$.*

Proof. Consider first an interval (k, n) such that $a_k a_n \neq 0$ but $a_j = 0$ for $k < j < n$. We claim that f has at least

$$n - k - \#(\text{sign changes in the pair } (a_k, a_n))$$

zeros on the open interval (k, n). Indeed, the number of zeros of f on this interval is at least $n-k-1$ in any case. This proves the claim if there is a sign change in the pair (a_k, a_n). If there is no sign change, that is $a_n a_k > 0$, then $f(n)f(k) = (-1)^{n-k}$. So the number of zeros of f on the interval (n, k) is of the same parity as $n-k$. But f has at least $n-k-1$ zeros on this interval, thus the total number of zeros is at least $n-k$. This proves our claim.

Now let a_k be the first and a_n the last non-zero term of our sequence. As the interval (k, n) is a disjoint union of the intervals to which the above claim applies, we conclude that the number of zeros of f on (k, n) is at least $(n-k)$ minus the number of sign changes of our sequence. On the rest of the interval $[0, N]$ our function has at least $N-n+k$ zeros, so the total number of zeros is at least N minus the number of sign changes. $\square$

Let u be a limit function of the subsequence $\{u_{m_k}\}$ with $m_k = [x_k]$. By Lemma 4, the function F has at least $arx_k - 2$ zeros on each interval $[x_k, (1+r)x_k]$, which implies that the Riesz measure μ of u satisfies

$$\mu([1, 1+r]) \geq ar.$$

This contradicts (9) and thus proves the implication b) $\longrightarrow$ a).

Proof of Theorem 2, $B \longrightarrow A$.

It is essentially the same as the previous proof. Proving by contradiction, we assume that B holds but there exists a function f of the form (4) with the property (2) which has an analytic continuation to I_Δ. Applying Lemma 1, we obtain an entire function F with properties (5) and (6). Then we consider subharmonic functions u_m and the limit functions u of this sequence. Using Lemmas 2, 3 and Grishin's lemma, we obtain the inequality (9) for the Riesz measure μ of u, exactly as in the proof of Theorem 1.

Now we notice that condition B of Theorem 2 means that the entire function F has zeros at some sequence of integers of maximum density at least $1-\Delta$. Denoting by $n(x)$ the number of zeros of F on $[0, x]$ and choosing a number $a \in (b, 1-\Delta)$, we obtain that there exist $r > 0$ and a sequence of integers $m_k \to \infty$ such that

$$n((1+r)m_k) - n(m_k) \geq arm_k.$$

This implies that for the limit function u of the sequence u_{m_k}, the Riesz measure μ satisfies $\mu([1, 1+r]) \geq ar$, which contradicts (9). This contradiction proves implication $B \longrightarrow A$ in Theorem 2.

Proof of implications a) $\longrightarrow$ b) of Theorem 1 and $A \longrightarrow B$ of Theorem 2. Suppose that a set Λ of positive integers does not satisfy b), B. We will construct power series f of the form (4) with real coefficients which has an immediate analytic continuation from the unit disc to the arc I_Δ. This will simultaneously prove the implications a) $\longrightarrow$ b) of Theorem 1 and $A \longrightarrow B$ of Corollary 1, as the number of sign changes of any sequence does not exceed the number of its non-zero terms.

Let $\Lambda' \subset \Lambda$ be a measurable set of density $\Delta' > \Delta$. Let S be the complement of Λ' in the set of positive integers. Then S is also measurable and has density $1 - \Delta'$.

Consider the infinite product

$$F(z) = \prod_{t \in S} \left(1 - \frac{z^2}{t^2}\right).$$

This is an entire function of exponential type with indicator $\pi(1-\Delta')|\sin\theta|$, and furthermore,

$$\log|F(z)| \geq \pi(1-\Delta')|\mathrm{Im}\, z| + o(|z|), \tag{10}$$

as $z \to \infty$ outside the set $\{z : \mathrm{dist}(z, S) \leq 1/4\}$. (See [10, Ch. II, Thm. 5] for this result.) Now we use the sufficiency part of Lemma 1, and define the coefficients of our power series by $a_m = (-1)^m F(m)$. Then we have all needed properties, in particular (2) follows from (10).

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