Midterm exam. Comments and solutions.

1 a). Answer: $c_n = \overline{c_{-n}}$. Proof: A function f is real iff $f(t) = \overline{f(t)}$, that is

$$\sum_{n=-\infty}^{\infty} c_n e^{int} = \sum_{n=-\infty}^{\infty} \overline{c_n} e^{-int}.$$

By uniqueness of a Fourier series, this happens if and only if $c_n = \overline{c_{-n}}$.

1 b). Answer: $c_n = -\overline{c_{-n}}$. Proof: A function f is pure imaginary iff $f(t) = -\overline{f(t)}$, the rest of the proof is similar to a).

1 c). Answer: $c_n = c_{-n}$. Proof: "Even" means $f(t) = f(-t)$, that is

$$\sum_{n=-\infty}^{\infty} c_n e^{int} = \sum_{n=-\infty}^{\infty} c_n e^{-int}.$$

By uniqueness sof a Fourier series, this happens iff $c_n = c_{-n}$.

1 d). Answer: $c_n = -c_{-n}$. Proof: same as c).

1 e). Answer: $c_n = 0$ for all odd n. Proof: Having π as a period means $f(t) = f(t + \pi)$, that is

$$\sum_{n=-\infty}^{\infty} c_n e^{int} = \sum_{n=-\infty}^{\infty} c_n e^{in(t+\pi)} = \sum_{n=-\infty}^{\infty} (-1)^n c_n e^{int}.$$

By uniqueness $c_n = (-1)^n c_n$, which means that $c_n = 0$ for all odd n.

Comment. This problem was intended to be the easiest: just apply the definitions. It turned out to be one of the hardest...

2. First solution. The system $(e^{int})_{n=-\infty}^{\infty}$ is complete and orthogonal in $L^2(-\pi, \pi)$ (This is a fundamental fact to remember!). This means that

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int},$$

and

$$\|f\|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2 \|e^{int}\|^2$$

(the Parseval identity). Computing the norms:

$$\|f\|^2 = \int_{-\pi}^{\pi} |f(t)|^2\,dt = 2\pi,$$

$$\|e^{int}\|^2 = \int_{-\pi}^{\pi} |e^{int}|^2\,dt = 2\pi.$$

Computing c_n:

$$\int_{-\pi}^{\pi} f(t)e^{-int}dt = c_n \int_{-\pi}^{\pi} \|e^{int}\|^2.$$

The integral in the LHS equals

$$\int_0^{\pi} (e^{-int} - e^{int})\,dt = -2i\int_0^{\pi} \sin ntdt = -4i/n,$$

when n is odd, and 0 otherwise. So $c_n = -2i/\pi$ when n is odd, and zero otherwise. Plugging this to Parseval's identity gives

$$\sum_0^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

2. Second solution, using only real numbers. The system $(\sin nt)_{n=1}^{\infty}$ is complete and orthogonal in $L^2(0,\pi)$. (This is also a very useful fact to remember). This means that

$$f(t) = \sum_{n=1}^{\infty} b_n \sin nt,$$

and

$$\|f\|^2 = \sum_{n=1}^{\infty} |b_n|^2 \|\sin nt\|^2.$$

Computing the norms:

$$\|f\|^2 = \int_0^{\pi} |f(t)|^2 dt = \pi,$$

and

$$\|\sin nt\|^2 = \int_0^{\pi} |\sin nt|^2 dt = \pi/2.$$

Computing the coefficients:

$$\int_0^{\pi} f(t)\sin nt\,dt = b_n\|\sin nt\|^2,$$

so $b_n = 4/(\pi n)$ for odd n and zero otherwise. Plugging to the Parseval identity gives

$$\sum_0^\infty \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

3. g is even, and h is odd, and $f(x) = g(x) + h(x)$. Replacing x by $-x$, we obtain $f(-x) = g(x) - h(x)$. Adding these two equations, we get $g(x) = (f(x) + f(-x))/2$, and subtracting them, $h(x) = (f(x) - f(-x))/2$. Thus

$$g(x) = (f(x) + f(-x))/2 = (xe^{-x} - xe^{x})/2 = -x \sinh x,$$

and

$$h(x) = (f(x) - f(-x))/2 = (xe^{-x} + xe^{x})/2 = x \cosh x.$$

Now, a_0 is the average of f over $(-\pi, \pi)$,

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} xe^{-x} dx \\ &= \frac{1}{2\pi} \left(-xe^{-x} \Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} e^{-x} dx \right) \\ &= \frac{1}{2\pi} (-2\pi \cosh \pi + 2 \sinh \pi) = (\sinh \pi)/\pi - \cosh \pi. \end{aligned}$$

Comment. Despite my hint, several students tried to find the Fourier coefficients. One succeeded. Which is an amazing achievement, given the time limitation. Others had the correct idea of "splitting f into even and odd parts", but nobody succeeded in doing this. This standard trick (splitting a function into an even and odd part) is useful to learn. Setting the integral for a_0 was a simple task, and most students did it, but rarely this integral was computed correctly.

4. Separation of variables gives $u = XT$, $XT' = X''T$,

$$\frac{T'}{T} = \frac{X''}{X} = -\lambda.$$

This leads to the boundary value problem

$$X'' + \lambda X = 0, \quad X'(0) = X'(\pi) = 0,$$

whose eigenvalues are $\lambda_n = n^2,\ n = 0, 1, 2, \ldots$ and the corresponding eigenfunctions are $X_n(x) = \cos nx$.

Then $T_n(t) = \exp(-n^2 t)$ and the general solution is

$$u(x,t) = \sum_{n=0}^{\infty} a_n \exp(-n^2 t) \cos nx.$$

Plugging $t = 0$ gives

$$\sin x = \sum_{n=0}^{\infty} a_n \cos nx,$$

where

$$a_n = \frac{2}{\pi} \int_0^{\pi} \sin x \cos nx \, dx, \quad \text{for} \quad n \geq 1,$$

and

$$a_0 = \frac{1}{\pi} \int_0^{\pi} \sin x \, dx = 2/\pi.$$

You may compute the integrals for full credit, but a crude answer to the last question can be given without this. Just notice that $|a_n| \leq 2$ for all n. The limit constant temperature is $2/\pi$. One percent of it is $\approx .006$. The first non-constant term of the Fourier series is $a_1 \exp(-t) \cos t$, so its absolute value is at most $2 \exp(-t)$. This will be less than .004 if $t > \ln 500 \approx 6.2$, so for example $t = 7$ is definitely enough, given the time limitation on the exam, and that calculators were not permitted. Even if you say $t = 8$ this is not considered a mistake. (Also notice that we gave a very generous allowance for the terms with $n \geq 2$, by assuming that their sum will not exceed .002).

Comments. The most common mistake was to confuse percent of the equilibrium temperature with the temperature itself. By the way, the Fourier coefficients for $n \geq 1$ are computed as follows:

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} \sin x \cos nx \, dx \\ &= \frac{1}{\pi} \int_0^{\pi} (\sin(1+n)x + \sin(1-n)x) dx \\ &= -\frac{1}{\pi} \left(\frac{1}{n+1} \cos(n+1)x + \frac{1}{1-n} \cos(1-n)x \right) \Big|_0^{\pi} \\ &= \frac{4}{\pi(1-n^2)}, \quad \text{when } n \text{ is even} \end{aligned}$$

and 0 otherwise. (The computation is a little bit different when $n = 1$, but this is irrelevant as $a_1 = 0$ anyway).

Now we can give a more precise answer to the last question, because we know that the first term is zero. The absolute value of the second term is at most $4e^{-4t}/(3\pi)$ and this is definitely less than .004 if $t > 2$. Calculator shows $t > 1.15$ will be enough.

5. We have

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx,$$

where $a_n = 0$ for all n because f is odd, while

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx \\ &= \frac{2}{\pi}\left(-\frac{x\cos nx}{n}\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\,dx\right) \\ &= \frac{2}{\pi n}(-\pi\cos n\pi) = 2(-1)^{n+1}/n. \end{aligned}$$

The sum of the series at the point $\pm\pi$ is $(f(\pi+) + f(\pi-))/2 = 0$. Using the hint, we multiply by 6 and integrate 2 times from 0 to obtain

$$x^3 = 12\sum_{1}^{\infty} \frac{(-1)^n}{n^3} \sin nx, \quad -\pi < x < \pi.$$