Practice problems for the final exam

April 27, 2021

1. Let f be a smooth function in $L^2(\mathbf{R})$. State the following properties in terms of Fourier transform $\hat{f}$:

a) f is real (takes real values for all real x),
b) f is even,
c) f is odd,
d) $f(x) = \overline{f(-x)}$, $x \in \mathbf{R}$

Solution.
a) We have
$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(s)e^{isx}ds,$$
so
$$\overline{f(x)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\hat{f}(s)}e^{-isx}ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\hat{f}(-s)}e^{isx}ds,$$
where we made the change of the variable $s \mapsto -s$. Since the function is uniquely defined by its Fourier transform, f will be real if and only if $\hat{f}(s) = \overline{\hat{f}(-s)}$.

b) We have
$$f(-x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(s)e^{-isx}dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(-s)e^{isx}ds.$$
So f is even if and only if $\hat{f}(s) = \hat{f}(-s)$, that is Fourier transform must be also even.

c) Same about odd.

e) We have
$$\overline{f(-x)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\hat{f}(s)}e^{isx}ds,$$
so for e), it is necessary and sufficient that $\hat{f} = \overline{\hat{f}}$ that is $\hat{f}$ is real.

2. Find $f \star g$, where $f(x) = x$ for $x > 0$ and $f(x) = 0$ for $x \le 0$, and $g(x) = e^x$ for $x < 0$ and $g(x) = 0$ for $x \ge 0$.

Solution.

$$(f \star g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy = \int_{-\infty}^{\min\{0,x\}} (x-y)e^y dy.$$

If $x < 0$, this is

$$\begin{aligned}\int_{-\infty}^{x} (x-y)e^y dy &= xe^x - \int_{-\infty}^{x} ye^y dy = xe^x - \int_{-\infty}^{x} yd(e^y) \\ &= xe^x - ye^y|_{-\infty}^{x} + \int_{-\infty}^{x} e^y dy = e^x.\end{aligned}$$

If $x \ge 0$, this is

$$\begin{aligned}\int_{-\infty}^{0} (x-y)e^y dy &= x - \int_{-\infty}^{0} ye^y dy \\ &= x - \int_{-\infty}^{0} yd(e^y) = x - ye^y|_{-\infty}^{0} + \int_{-\infty}^{0} e^y dy \\ &= x + 1.\end{aligned}$$

So

$$(f \star g)(x) = \begin{cases} e^x, & x < 0, \\ x+1, & x \ge 0. \end{cases}$$

3. How many eigenvalues of the Sturm-Liouville problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(1) + y'(1) = 0$$

satisfy $-10 < \lambda_j < 10$?

Solution. If $\lambda > 0$ the general solution is

$$y(x) = a\cos(x\sqrt{\lambda}) + b\sin(x\sqrt{\lambda}).$$

since $y(0) = 0$ we have $a = 0$, and can take $b = 1$, so

$$y(1) + y'(1) = \sin\sqrt{\lambda} + \sqrt{\lambda}\cos\sqrt{\lambda} = 0,$$

and we have the equation

$$\tan\sqrt{\lambda} = -\sqrt{\lambda},$$

which has to be solved for $0 < \sqrt{\lambda} < \sqrt{10} \approx \pi$. It has one solution on this interval.

For $\lambda < 0$, the general solution is

$$y(x) = a\cosh(x\sqrt{-\lambda}) + b\sinh(x\sqrt{-\lambda}),$$

and again $a = 0$ because $y(0) = 0$, and we have to solve the equation

$$\tanh\sqrt{-\lambda} = -\sqrt{-\lambda}.$$

This has no positive solutions.

Finally for $\lambda = 0$, the general solution is $y(x) = ax + b$, since $y(0) = 0$ we have $b = 0$, and the second boundary condition gives $a + a = 0$, so $a = 0$. So 0 is not an eigenvalue.

Thus the answer is: one.

4. Find a bounded solution of the Laplace equation in the region $\{(x, y) : -\infty < x < \infty, y > 0\}$ with the boundary conditions

$$u(x, 0) = \begin{cases} 1, & |x| < 1, \\ 0, & |x| > 1. \end{cases}$$

Solution. Poisson's formula for the half-plane gives

$$u(x) = \frac{y}{\pi} \int_{-1}^{1} \frac{dt}{(x-t)^2 + y^2} = \frac{1}{\pi} \left(\arctan \frac{y}{x-1} - \arctan \frac{y}{x+1} \right).$$

5. Bessel's function $y = J_0(x)$ satisfies the differential equation

$$xy'' + y' + xy = 0.$$

a) Write a differential equation for the function $w(x) = J_0(\sqrt{x})$.

b) Consider the partial differential equation

$$u_{tt} = (xu_x)_x, \quad t > 0 \quad 0 < x < 1,$$

with the boundary conditions $u(1,t) = 0$ and $|u(0,t)| < \infty$. Separate the variables, find the eigenvalues of the related eigenvalue problem and write the general solution satisfying the boundary conditions as a series of the form

$$\sum_n f_n(t) g_n(x).$$

f_n and g_n must be expressed in terms of exponentials, Bessel functions and zeros of Bessel functions. Hint: use part a).

Solution. a) Let $y = J_0$. Then $y(x) = w(x^2)$, $y'(x) = 2xw'(x^2)$ and $y''(x) = 2w'(x^2) + 4x^2w''(x^2)$. Plugging all this to the differential equation for J_0 we obtain

$$4x^3w''(x^2) + 4xw'(x^2) + xw(x^2) = 0.$$

Dividing on $4x$, and then replacing x^2 by x we obtain

$$xw'' + w' + w/4 = 0. \tag{1}$$

So solutions of this equation which are finite at 0 are of the form $cJ_0(\sqrt{x})$, where c is a constant.

b) Write the equation in the form $u_{tt} = xu_{xx} + u_x$; substituting $u(x,t) = f(t)g(x)$ to separate the variables, we obtain

$$f''g = xfg'' + fg',$$

$$\frac{f''}{f} = x\frac{g''}{g} + \frac{g'}{g} = -\lambda$$

where λ is a constant. In x-variable, we have the boundary value problem

$$xg'' + g' + \lambda g = 0 \tag{2}$$

and the boundary conditions that $g(0)$ is finite and $g(1) = 0$.

To obtain the equation (1) that we solved in part a), we do the same that we did with Bessel functions, that is look for g in the form $g(x) = w(4\lambda x)$. Then $g' = 4\lambda w'(4\lambda x)$, $g'' = 16\lambda^2 w''(4\lambda x)$ and thus w will solve (1). Thus we obtained the solution of (2) in the form

$$g(x) = w(4\lambda x) = J_0(2\sqrt{\lambda x}).$$

Substituting the boundary condition $g(1) = 0$ we obtain that $2\sqrt{\lambda} = x_k$, a zero of J_0, so the eigenvalues are

$$\lambda_k = x_k^2/4, \quad k = 1, 2, 3, \ldots$$

and eigenfunctions are

$$g_k(x) = J_0(x_k\sqrt{x}).$$

It remains to solve for the time component:

$$f'' + \lambda f = 0.$$

since all λ_k are positive we have

$$f_k(t) = a_k \cos(\sqrt{\lambda_k}t) + b_k \sin(\sqrt{\lambda_k}t) = a_k \cos(x_k t/2) + b_k \sin(x_k t/2).$$

So the answer is

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos(x_n t/2) + b_n \sin(x_n t/2)\right) J_0(x_k\sqrt{x}).$$

6. Suppose that a function f of one real variable x, and its Fourier transform $\hat{f}$ are known. Express Fourier transforms of the following functions g in terms of f and $\hat{f}$. You may use convolution in your answer.

a) $g(x) = xf(-2x)$,

b) $g(x) = f''(x) + xf(x)$,

c) $g(x) = f^2(x-2)$,

d) $g(x) = \hat{f}(x)$,

e) $g(x) = e^{-x^2} f(x)$.

f) $g(x) = \overline{f(x)}$.

Answers.

a) $(-i/4)\hat{f}'(-s/2)$

b) $-s^2\hat{f}(s) + if'(s)$

c) $\dfrac{1}{2\pi}e^{-2is}(\hat{f} \star \hat{f})(s)$

d) $2\pi f(-x)$

e) $\sqrt{\pi}e^{-s^2/4} \star \hat{f}(s)$.

f) $\overline{\hat{f}(-s)}$.

7. For the function

$$f(x) = \begin{cases} 1, & |x| < 1, \\ 0 & \text{otherwise} \end{cases}$$

a) Compute $f \star f$.

b) Find the Fourier transform of $f^{\star 100} = f \star f \star \ldots \star f$, the convolution of f with itself 100 times.

Answers.

a)

$$(f \star f)(x) = \begin{cases} 2 - 2x, & |x| < 2 \\ 0, & \text{otherwise} \end{cases}$$

b) $\dfrac{2^n \sin^n s}{s^n}$.

8. Find five smallest eigenvalues λ of the Laplace equation

$$\Delta u + \lambda u = 0$$

in the square

$$\{(x, y) \in \mathbf{R}^2 : 0 < x < 1,\ 0 < y < 1\}$$

with the boundary conditions

$$u_x(0) = u_x(1) = 0, \quad u_y(0) = y_y(1) = 0.$$

Find dimensions of eigenspaces corresponding to these eigenvalues.

Solution. To separate the variables we write $u = XY$, then

$$\frac{X''}{X} + \frac{Y''}{Y} + \lambda = 0.$$

Since each summand in the LHS is a function of one variable, all these summands must be constants; denote then by $-p$ and $-q$. Then

$$X'' + pX = 0, \quad X(0) = 0, X'(1) = 0.$$

The general solution is

$$X(x_1) = a\cos(\sqrt{p}x) + b\sin(\sqrt{q}x).$$

the boundary conditions imply $b = 0$ and $\sin\sqrt{p} = 0$, so $p = \pi^2 m^2$, and similarly $q = \pi^2 n^2$, where $m, n = 0, 1, 2, \ldots$. So

$$\lambda_{m,n} = \pi^2(m^2 + n^2),$$

and the smallest eigenvalues are 0 (one-dimensional eigenspace) π^2 (two-dimensional eigenspace) $2\pi^2$ (one dimensional eigenspace) $4\pi^2$ (two-dimensional eigenspace) and $5\pi^2$ (two-dimensional eigenspace).