Practice problems for the final exam

April 7, 2021

1. Let f be a smooth function in $L^2(\mathbf{R})$. State the following properties in terms of Fourier transform $\hat{f}$:

a) f is real (takes real values for all real x),

b) f is even,

c) f is odd,

d) $f(x) = \overline{f(-x)},\ x \in \mathbf{R}$

2. Find $f \star g$, where $f(x) = x$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$, and $g(x) = e^x$ for $x < 0$ and $g(x) = 0$ for $x \geq 0$.

3. How many eigenvalues of the Sturm-Liouville problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(1) + y'(1) = 0$$

satisfy $-10 < \lambda_j < 10$?

4. Find a bounded solution of the Laplace equation in the region $\{(x, y) : -\infty < x < \infty, y > 0\}$ with the boundary conditions

$$u(x, 0) = \begin{cases} 1, & |x| < 1, \\ 0, & |x| > 1. \end{cases}$$

5. Bessel's function $y = J_0(x)$ satisfies the differential equation

$$xy'' + y' + xy = 0.$$

a) Write a differential equation for the function $w(x) = J_0(\sqrt{x})$.

b) Consider the partial differential equation

$$u_{tt} = (xu_x)_x, \quad t > 0 \quad 0 < x < 1,$$

with the boundary conditions $u(1,t) = 0$ and $|u(0,t)| < \infty$. Separate the variables, find the eigenvalues of the related eigenvalue problem and write the general solution satisfying the boundary conditions as a series of the form

$$\sum_n f_n(t) g_n(x).$$

f_n and g_n must be expressed in terms of exponentials, Bessel functions and zeros of Bessel functions. Hint: use part a).

6. Suppose that a function f of one real variable x, and its Fourier transform $\hat{f}$ are known. Express Fourier transforms of the following functions g in terms of f and $\hat{f}$. You may use convolution in your answer.

a) $g(x) = xf(-2x)$,

b) $g(x) = f''(x) + xf(x)$,

c) $g(x) = f^2(x-2)$,

d) $g(x) = \hat{f}(x)$,

e) $g(x) = e^{-x^2} f(x)$.

f) $g(x) = \overline{f(x)}$.

7. For the function

$$f(x) = \begin{cases} 1, & |x| < 1, \\ 0 & \text{otherwise} \end{cases}$$

a) Compute $f \star f$.

b) Find the Fourier transform of $f^{\star 100} = f \star f \star \ldots \star f$, the convolution of f with itself 100 times.

8. Find five smallest eigenvalues λ of the Laplace equation

$$\Delta u + \lambda u = 0$$

in the square

$$\{(x, y) \in \mathbf{R}^2 : 0 < x < 1,\ 0 < y < 1\}$$

with the boundary conditions

$$u_x(0) = u_x(1) = 0, \quad u_y(0) = y_y(1) = 0.$$

Find dimensions of eigenspaces corresponding to these eigenvalues.