**Math 452/525 final exam, fall 2020**

NAME:

1. Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}.$$

(The answer should be a real number).

ANSWER: $(\pi/\sinh\pi - 1)/2$

2. Find the radii of convergence of the following series:

a) $\sum_{n=1}^{\infty} \frac{5^n}{n} z^{2n}$,

ANSWER: $1/\sqrt{5}$

b) $\sum_{n=1}^{\infty} \frac{n^n}{n!} z^n$,

ANSWER: $1/e$

c) $\sum_{n=1}^{\infty} \frac{n!}{10^n} z^n$,

ANSWER: 0

d) $\sum_{n=0}^{\infty} a_n z^n = \frac{e^z}{z^2 + z + 2}$,

ANSWER: $\sqrt{2}$

e) $\sum_{n=0}^{\infty} a_n z^n = \frac{\text{Log}\,(z+2)}{z+1}$.

ANSWER: 2.

3. Evaluate the integral

$$\int_0^\infty \frac{\cos(x\sqrt{2})}{x^4+1}\,dx.$$

(The answer should be a real number).

ANSWER: $\pi 2^{-3/2} e^{-1}(\cos 1 + \sin 1)$

4. Evaluate the integral

$$\int_{|z|=2} (z^3 + z) \cos\left(\frac{1}{z}\right) dz,$$

where the circle is described counterclockwise.

ANSWER: $-11i\pi/12$.

5. For which complex values of $a$ the equation

$$\cot z = a$$

has no complex solutions?

ANSWER: $a = i$ and $a = -i$.

6. a) Find the conformal map $f$ of the region

$$\{z : \operatorname{Re} z > 0,\ \operatorname{Im} z > 0,\ |z| < 1\}$$

onto the upper half-plane, such that

$$f(i) = 0, \quad f(0) = 1, \quad f(1) = \infty.$$

b) Find $f((1+i)/2)$.

ANSWER:

$$f(z) = (J(z^2)+1)/(J(z^2)-1) = \left(\frac{z^2+1}{z^2-1}\right)^2, \quad f((1+i)/2) = (-7+24i)/25.$$

Solution 1. $z^2$ maps our region onto the upper half of the unit disk. Then Joukowski function $J(z) = (z+1/z)/2$ maps the upper half of the unit disk onto the lower half-plane. Then plugging the values $(i, 0, 1)$ to $J(z^2)$ we obtain $(-1, \infty, 1)$ and it remains to send $(-1, \infty, 1)$ to $(0, 1, \infty)$ which is done by the linear-fractional transformation

$$\frac{z+1}{z-1}.$$

So the answer is

$$\frac{J(z^2)+1}{J(z^2)-1} = \left(\frac{z^2+1}{z^2-1}\right)^2.$$

Solution 2. The first step $z^2$ is the same, but if you do not remember the Joukowski function, you can treat the upper half-disk as a "moon", and aply

$$\frac{z+1}{z-1}$$

to it (to send the corners $(-1, 1)$ to $(0, \infty)$. Under this map, the image of the "moon" is the 3-d quadrant. It is sent to the upper half-plane by another squaring, which results in the function

$$\left(\frac{z^2+1}{z^2-1}\right)^2,$$

which happens to give the prescribed correspondence of the three boundary points.

7. How many solutions does the equation

$$e^{z-2} = z^2$$

have in the unit disk $\{z : |z| < 1\}$?

ANSWER: two.

8. Does there exist a real function $u(x, y)$ such that the function

$$f(x+iy) = u(x, y) + i(x - x^3 + 3xy^2)$$

is analytic in the whole plane?

If the answer is "no", explain why; if the answer is "yes", find this function $u$.

ANSWER: $u(x, y) = -y - y^3 + 3x^2y + c$.