Math 425, Fall 2015, final exam

NAME:

1. How many roots does the equation

$$z + e^{-z} = 2$$

have in the right half-plane?

2. Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

3. Evaluate the integral

$$\int_0^\infty \frac{\cos x}{(x^2+1)^2} dx.$$

4. Show that there exists a unique power series of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

satisfying the equations

$$f'(z) = f(z/2), \quad f(0) = 1,$$

and find the radius of convergence of this series.

5. Find and classify the isolated singularities in $\mathbf{C}$ of the following functions at the specified points. For poles also find the order.

$$a) \qquad \frac{1}{(2\cos z - 2 + z^2)^2}, \qquad z = 0.$$

$$b) \qquad \cot(1 - 1/z), \qquad z = 1.$$

$$c) \qquad \frac{\text{Log} z}{(z-1)^2}, \qquad z = 1.$$

$$d) \qquad \exp(z) - 1/z, \qquad z = \infty.$$

$$e) \qquad \exp\left(\frac{1}{z^2 - 1}\right), \qquad z = 1.$$

6. Determine the first three terms (of the smallest degree) in the Laurent expansion of

$$\frac{1}{e^z - 1} \quad \text{in} \quad \{z : 0 < |z| < 2\pi\}.$$

7. Let f be an analytic function in a disk $|z| < R,\ R > 1$, and suppose that $|f(z)| \leq 1$ for all z on the unit circle.

a) Use Cauchy's formula to prove that $|f'(0)| \leq 1$.

Please give a complete argument.

b) For which functions can equality in a) happen?

8. a) Find a conformal map of the region

$$D = \{z : |z| < 1, \operatorname{Im} z > 0\}$$

onto the upper half-plane, and the inverse map.

b) Solve the following Dirichlet problem:

$$\Delta u = 0, \quad \text{in} \quad D,$$

and u is bounded in D,

$$u(x) = 1, \quad -1 < x < 1,$$

$$u(e^{it}) = 0, \quad 0 < t < \pi.$$