Math 511, Final exam solved, fall 2019

NAME:

Problems 1-4 are multiple choice: just circle the letters, no partial credit.

Problems 5-9 are partial credit, but please write your **answer**, if you obtain one, next to the problem.

1. Circle the letters corresponding to the statements which are true for all square matrices A, B of the same size.

A. If v is an eigenvector of A then v is also an eigenvector of e^A.

B. $e^{A+B} = e^A e^B$.

C. If A and B are similar then $\operatorname{tr} A = \operatorname{tr} B$.

D. If A is non-singular then it is diagonalizable.

E. If $A^{2019} = 0$ then A is singular.

Ans.: A,C,E.

Solution. A. If $Av = \lambda v$ then $A^m = \lambda^m v$, so

$$e^A v = \left(\sum_{m=0}^{\infty} \frac{A^m}{m!} \right) v = \left(\sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \right) v = e^{\lambda} v.$$

B. Not true. The LHS does not change if we interchange A and B while the right side can change.

C. True, since similar matrices have the same characteristic polynomial:

$$\det (CAC^{-1} - \lambda I) = \det (C(A - \lambda I)C^{-1}) = \det (A - \lambda I).$$

D. Not true. Counterexample is a Jordan cell with non-zero eigenvalue.

E. True. If $A^{2019}x = 0,\ x \neq 0$, then $A^{2018}x$ belongs to $N(A)$. If $A^{2018}x = 0$ then $A^{2017}x$ belongs to $N(A)$, and so on. Eventually we will find a non-zero vector in $N(A)$. This means that A is singular.

2. Circle the letters which correspond to the statements which are true for all square 5×5 matrices A, B, C.

A. $\det(-A) = -\det(A)$.

B. $\det(3A) = 3\det(A)$.

C. $\det(A+B) = \det(A) + \det(B)$.

D. $\det(ABC) = \det(A)\det(B)\det(C)$.

E. Determinant of A does not change if the rows A are rearranged in the opposite order.

Ans.: A,D,E.

Solution. A. True. Multiplying A on -1 is the same as multiplying each row on -1, so determinant is multiplied on $(-1)^5 = -1$.

B. No. The correct formula is $\det(3A) = 3^5 \det(A)$.

C. No. For example, it is easy to write I, whose determinant is 1, as a sum of two diagonal matrices with zeros and ones on the main diagonal; their determinants are 0.

D. True. In general $\det(AB) = \det(A)\det(B)$, this is a theorem which was proved in class. Applying it twice we obtain the formula in D.

E. True. This depends on the sign of the permutation $(n, n-1, \ldots, 1)$. For $n = 5$ this permutation has 2 transpositions so it is even.

3. Circle the letters which correspond to true statements for all square matrices of the same size:

A. If A and B are Hermitian matrices then $A + B$ is Hermitian.

B. If A and B are Hermitian then AB is Hermitian.

C. If A and B are unitary then $A + B$ is unitary.

D. If A and B are unitary then AB is unitary.

E. If A is Hermitian and B is unitary then $B^{-1}AB$ is defined and is Hermitian.

Ans.: A,D,E.

Solutions. A. True. This was proved in class. It follows from the rule $(A + B)^* = A^* + B^*$.

B. False. $(AB)^* = B^*A^*$. This is equal to A^*B^* only when A and B commute.

C. False. Take $A = I,\ B = -I$. They are unitary but their sum is 0 not unitary.

D. True. $(AB)(AB)^* = ABB^*A^* = I$.

E. True. This is defined because unitary matrices are invertible, and $B^{-1} = B^*$. Thus

$$(B^{-1}AB)^* = (B^*AB)^* = B^*A^*B = B^{-1}AB.$$

4. Suppose that A is a real symmetric negative definite matrix of size 4×4, that is $x^T Ax < 0$ for all $x \neq 0$. What conclusions can be made from this ? Circle the corresponding letters.

A. A is non-singular

B. All eigenvalues of A are strictly negative.

C. Determinant of A is negative.

D. All upper left minors of A are negative.

E. No row exchanges are required when bringing A to the upper triangular form by row operations.

Ans.: A,B,E.

Solution. A. True. $x^T Ax < 0$ for all $x \neq 0$ means that the signature consists of n minuses.

B. True. By the spectral theorem, A is congruent to the diagonal matrix with eigenvalues on the main diagonal.

C. False. Determinant is the product of eigenvalues. Since they are all negative and there are 4 of them, the determinant is positive.

D. False. For the same reason as C.

E. True. $-A$ is positive definite. For positive definite matrix, no row exchanges are required (this was proved in class). So for A they are also not required.

5. For the matrix

$$\begin{pmatrix} 2 & 2 & -2 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix},$$

find the Jordan form and a Jordan basis.

Solution. The characteristic polynomial is

$$-\lambda(\lambda-1)^2.$$

To obtain it, expand $\det(A-\lambda I)$ along the second row. For $\lambda_1 = 0$ we obtain an eigenvector $v_1 = (1,0,1)^T$. For $\lambda_2 = 1$ we obtain an eigenvector $v_1 = (2,0,1)^T$. Since the eigenspace of λ_2 is one-dimensional, and λ_2 is the root of multiplicity 2, there must be a generalized eigenvector, So the Jordan form must be

$$J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solving $(A-I)v_3 = v_2$ we find $v_3 = (0,1,0)^T$. These v_1, v_2, v_3 form a Jordan basis.

6. Find e^{At} for the matrix

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

(Exponential is defined as a matrix solution $X(t)$ of the differential equation

$$\frac{d}{dt}X = AX, \quad \text{such that} \quad X(0) = I.)$$

Solution. The characteristic equation of A is

$$\lambda^2 - 2\lambda + 2 = 0.$$

Taking $\lambda_1 = 1 + i$ we find an eigenvector $v_1 = (1, i)^T$. To this eigenvector corresponds a solution of the differential equation

$$e^{(1+i)t}\begin{pmatrix} 1 \\ i \end{pmatrix} = e^t(\cos t + i \sin t)\begin{pmatrix} 1 \\ i \end{pmatrix} = e^t\left(\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + i\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}\right).$$

So a real fundamental matrix is

$$\Phi(t) = e^t\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

Since $\Phi(0) = I$ it equals e^{At}.

7. Find the signature of the quadratic form

$$xy + yz + xz.$$

Solution. Set $x = u + v,\ y = u - v.$ Then

$$\begin{aligned}
& xy + yz + xz \\
= \ & u^2 - v^2 + uz - vz + uz + vz = u^2 - v^2 + 2uz \\
= \ & (u^2 + 2uz + z^2) - z^2 - v^2 = (u + z)^2 - z^2 - v^2,
\end{aligned}$$

so the signature is $(+, -, -)$.

8. Evaluate the determinant

$$\begin{vmatrix} x & y & y & y & y \\ y & x & y & y & y \\ y & y & x & y & y \\ y & y & y & x & y \\ y & y & y & y & y \end{vmatrix}.$$

Solution. Subtract the last column from each of the rest. We obtain an upper triangular matrix whose determinant is $(x-y)^4y$.

9. The first and second columns of a rotation matrix A are

$$\begin{pmatrix} 1/3 \\ 2/3 \\ -2/3 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 2/3 \\ 1/3 \\ 2/3 \end{pmatrix}.$$

a) Find the third column of A.

b) How many solutions does this problem have?

c) If two columns a and b are given, what are the conditions on a and b for this problem to be solvable?

Solution. The two given columns are orthonormal. The third columns must be orthogonal to them, so for its coordinates we have a system of two equations:

$$\begin{aligned} x_1 + 2x_2 - 2x_3 &= 0 \\ 2x_1 + x_2 + 2x_3 &= 0. \end{aligned}$$

This system has one dimensional set of solutions which is spanned by the vector $(-2, 2, 1)^T$. Normalizing it we obtain two possible third columns $\pm(-2/3, 2/3, 1/3)^T$. To choose one of them, we need to use the condition that determinant of an orthogonal matrix must be 1. Since

$$\begin{vmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{vmatrix} = -27,$$

we have to choose the minus sign. So the third column is $(2/3, -2/3, -1/3)^T$. This answers a).

An analysis of this solution shows that the answer to b) is "one".

For c), we must have $a^T b = 0$, $\|a\| = \|b\| = 1$. If these conditions are satisfied, then the system of equations obtained from the condition of the orthogonality of the third column will have one-dimensional set of solutions, so there will be two unit vectors satisfying this system, and one of them will give a positive determinant.