Math 525, Final exam. Fall 2021

NAME:

1. Solve the equation

$$\tan z = 2i.$$

Write all solutions and make a picture of them.

Solution. The equation is

$$\frac{\sin z}{\cos z} = 2i$$

Let $w = e^{iz}$, then our equation becomes

$$\frac{w - w^{-1}}{i(w + w^{-1})} = 2i,$$

or

$$3w^2 + 1 = 0,$$

so $w_{1,2} = \pm i/\sqrt{3}$. Now

$$iz = \log w = \log|w| + i \arg w = -(1/2)\mathrm{Log}\, 3 \pm i\pi/2 + 2\pi i n,$$

and

$$z = (i/2)\mathrm{Log}\, 3 \pm \pi/2 + 2\pi n.$$

To make a good picture, notice that $\mathrm{Log}\, 3 > 0$, and that the sequence has period π.

2. Evaluate the integral

$$\int_0^\infty \frac{\cos z}{z^2+1} dz.$$

Solution. Denote our integral by I. Then

$$I = \frac{1}{2} \mathrm{Re} \int_{-\infty}^{\infty} \frac{e^{iz}}{z^2+1} = \frac{1}{2} \mathrm{Re}\, 2\pi i \,\mathrm{res}_i,$$

and

$$\mathrm{res}_i \frac{e^{iz}}{(z-i)(z+i)} = \frac{e^{-1}}{2i}.$$

So we obtain $I = \pi/(2e)$.

3. a) Find a conformal maps f of the region

$$\{z = x + iy : x > 0, y > 0, x^2 + y^2 < 1\}$$

onto the upper half-plane.

b) Is it possible to find such a conformal map with the additional properties $f(1) = 0,\ f(i) = 1,\ f(0) = \infty$? If your answer is positive, find it, if negative, explain why.

Solution. The region is the intersection of the unit disk with the first quadrant. Function z^2 maps it onto the upper half of the unit disk. To send the corners of the upper half of the unit disk to $(0, \infty)$, apply

$$\frac{z+1}{z-1},$$

which maps the upper half of the unit disk onto the 3-d quadrant. Then square the result to obtain the upper half-plane. So an answer to a) can be

$$f(z) = \left(\frac{z^2+1}{z^2-1}\right)^2.$$

To answer b), notice that $(1, i, 0)$ are in the order of positive orientation of our region (the region stays on the left), and f sends $(1, i, 0)$ to $(\infty, 0, 1)$ (in this order). The points $(0, 1, \infty)$ are in the same order on the boundary of the upper half-plane, therefore there is a fractional-linear transformation of the upper half-plane onto itself that sends $(\infty, 0, 1)$ to $(0, 1, \infty)$. So the answer to b) is "yes", and the required transformation is

$$\frac{1}{1-z}.$$

Combining all pieces we obtain

$$\frac{1}{1 - \left(\frac{z^2+1}{z^2-1}\right)^2} = -\frac{(z^2-1)^2}{4z^2}.$$

4. How many solutions does the equation

$$e^z = z^2 - 2$$

have in the left half-plane?

Solution. Apply Rouche's theorem to a large half-disk of the form

$$\{z : |z| = R, \operatorname{Re} z < 0\}.$$

Our equation is equivalent to

$$e^z - z^2 + 2 = 0.$$

We have $|e^z| \leq 1$ in the closed left half-plane, while

$$| - z^2 + 2| \geq |z^2| - 2 = R^2 - 2 \to +\infty, \quad R \to +\infty,$$

so the function $z^2 - 2$ is large on the half-circle, while on the imaginary axis, $z = it$,

$$|z^2 - 2| = |(iy)^2 - 2| = | - y^2 - 2| = y^2 + 2 > 1.$$

So everywhere on the boundary $|e^z| < |z^2 - 2|$, and the equation has the same number of solutions in the region as $z^2 - 2 = 0$, that is one solution.

5. Evaluate the integral

$$\int_{|z|=4} \frac{e^z}{\sin z} dz.$$

Solution. The function has 3 poles in the disk $|z| < 4$. (The poles are at πn and $|\pi n| < 4$ for $n = 0, 1, -1$.) All these poles are simple, and the residues are

$$\mathrm{res}_0 \frac{e^z}{\sin z} = 1,$$

$$\mathrm{res}_\pi \frac{e^z}{\sin z} = -e^\pi,$$

and

$$\mathrm{res}_{-\pi} \frac{e^z}{\sin z} = -e^{-\pi}.$$

So by the Residue theorem, the integral is equal to

$$2\pi i(1 - e^\pi - e^{-\pi}) = 2i\pi(1 - 2\cosh \pi).$$

6. Describe and sketch the set

$$\{z : \left|e^{z^2}\right| = 2\}.$$

Solutiona Let $z = x + iy$. Then

$$|e^{z^2}| = e^{\mathrm{Re}\,(x+iy)^2} = e^{x^2-y^2} = 2 = e^{\mathrm{Log}\,2}$$

defines a hyerbola

$$x^2 - y^2 = \mathrm{Log}\,2.$$

7. a) Find the radius of convergence of the series

$$f(z) = \sum_{1}^{\infty} \frac{2^n}{n} z^n.$$

b) Find $f(z)$ explicitly (it is an elementary function).

Solution. Since

$$\left(\frac{2^n}{n}\right)^{1/n} \to 2, \quad n \to \infty,$$

the radius of convergence is $1/2$.

To answer b), differentiate the series:

$$f'(z) = \sum_{1}^{\infty} 2^n z^{n-1} = \frac{2}{1-2z}.$$

Integrating, we obtain

$$f(z) = -\mathrm{Log}\,(1-2z) + c,$$

and plugging $z = 0$ we obtain $c = 0$.