April, 2000

Name: ______________________________

1. How many roots do these equations have in the specified regions?

   a) $z^4 - 3z + 1 = 0$ in $|z| < 1$,

   b) $z = 2 - e^{-z}$ in the right half-plane; on the positive ray.

   c) $z^6 - 6z + 10$ in $|z| > 1$.

2. Evaluate the integrals

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 - 2ix - 2} dx, \quad \int_0^{\infty} \frac{\operatorname{Log} x}{(x+1)^2} dx, \quad \int_0^{\infty} \frac{1 - \cos 2x}{x^2} dx.$$

3. Find the sum of the series

$$\sum_{n=1}^{\infty} (-1)^n n^{-2},$$

   by integrating $z^{-2} \csc \pi z$ over an appropriate countur.

4. Find the Laurent series expansions:

   a) of $1/(z-2)$ in $1 < |z-3| < \infty$,

   b) of $z^2 \sin \pi \frac{z+1}{z}$ in $0 < |z| < \infty$.

5. Find and classify all isolated singularities of these functions:

$$\frac{\sin \pi z^2}{\sin \pi z}, \quad \exp \tan z, \quad \frac{1}{e^z + 1}.$$

6. Find the radius of convergence of these power series

$$\frac{\operatorname{Log} z}{z - 1} = \sum_{n=0}^{\infty} a_n (z-3)^n,$$

$$\frac{\operatorname{Log} z}{z - 1} = \sum_{n=0}^{\infty} b_n (z+1+i)^n,$$

$$\sum_{n=0}^{\infty} 2^n z^{n!}.$$

7. Expand into Laurent series:

$$(z^2-1)^{-1} \text{in} \quad |z|>1,$$

$$(z^2-1)^{-1} \text{in} \quad 0<|z+1|<2,$$

$$z^2\sin(1/z) \text{in} \quad |z|>0.$$

8. Find a harmonic conjugate to

$$e^x(x\cos y - y\sin y).$$

9. Yes or no:

a) $\text{Log}|z|$ is harmonic in $0<|z|<\infty$,

b) $\text{Log}|z|$ is the real part of some analytic function in $0<|z|<\infty$,

c) $\int e^{1/z^2}dz$ is path independent in $0<|z|<\infty$,

d) $\int ze^{1/z^2}dz$ is path independent in $0<|z|<\infty$,

e) $|\cos z|$ for $|z|\le 1$ has maximal value at the point 0.

f) if P is a polynomial of degree at least 2, whose all zeros belong to the unit disc, then $\int_{|z|=2}(1/P(z))dz=0$.