Math 425/525 Fall 2002 final exam

NAME:

1. a) What is the multiplicity of the pole of the function

$$\frac{1}{(\sin(2z) - 2z)^2}$$

at $z = 0$?

b) What is the residue at this pole?

Solutions a).

$$\sin(2z) - 2z = 2z - (2z)^3/6 + \ldots - 2z = -(2z)^3/6 + \ldots,$$

therefore the multiplicity of the pole is $2 \times 3 = 6$.

b) The residue is zero since the function is even.

2. Find the radii of convergence of the following series. (Write your answer next to each series. No partial credit for this problem.)

$$a) \quad \sum_{n=0}^{\infty} \frac{z^{2n}}{2^n}, \qquad R = \sqrt{2}$$

$$b) \quad \sum_{n=0}^{\infty} 2^n z^{n^2}, \qquad R = 1$$

$$c) \quad \sum_{n=1}^{\infty} z^{n!}, \qquad R = 1$$

$$d) \quad \sum_{n=0}^{\infty} c_n (z-1/2)^n = \frac{z-1}{z^4-1}, \qquad R = \sqrt{5}/2$$

$$e) \quad \sum_{n=1}^{\infty} \frac{n!}{2^{n^2}} z^n, \qquad R = \infty.$$

Solution for c). $z^4 - 1 = (z-1)(z+1)(z+i)(z-i)$, so $z = 1$ is removable, and since we expand at $1/2$, the radius of convergence is the distance from $1/2$ to the closest pole. The closest pole is i (or $-i$). the distance is $\sqrt{1/4+1} = \sqrt{5}/2$. To make sure that it is the closest compare it with the pole at -1. Indeed, we have $\sqrt{5}/2 < 3/2$ by squaring both sides.

3. Find all isolated singularities of this function in $\overline{\mathbf{C}}$, and classify them into removable, poles and essential. For each pole, tell its multiplicity.

$$\frac{\sin(\pi/z)}{(4z^2-1)^2}.$$

Solution. $z = 0$ is an essential singularity, and $z = \infty$ is removable. Other possible singularities can be poles at $\pm 1/2$. At these points the denominator has zeros of multiplicity 2 and numerator has zeros of multiplicity 1. Therefore both poles at $\pm 1/2$ are simple (of multiplicity 1).

4. Evaluate the integral

$$\int_0^\infty \frac{x^{1/3}}{(1+x)^2} dx.$$

Your answer should be a real number.

Solution. Let $f(z) = z^{1/3}/(1+z)^2$, where the branch used is with $0 < \arg z < 2\pi$. We integrate f over the contour consisting of a circle $|z| = R$ and two rays $\{z = t : 0 < t < R\}$ and $\{z = te^{2\pi i} : 0 < t < R\}$ oriented in the opposite direction. So if our integral is I, then the residue theorem gives

$$I(1 - e^{2\pi i/3}) = 2\pi i \operatorname{res}_{-1} f.$$

Computing the residue at the second order pole

$$\operatorname{res}_{-1} f(z) = \lim_{z \to -1} (d/dz) z^{1/3} = (1/3) z^{-2/3}|_{z=-1} = e^{-2\pi i/3}/3.$$

Now compute I:

$$I = \frac{2\pi i e^{-2\pi i/3}}{3(1 - e^{2\pi i/3})} = \frac{2\pi}{3\sqrt{3}}.$$

5. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1} dx.$$

Solution.

$$I = \text{Rea} \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1} = \text{Rea} \left(2\pi i \, \text{res}_i \frac{e^{iz}}{z^2+1} \right) = \frac{\pi}{e}.$$

6. a) Does there exist a linear-fractional map

$$f(z) = \frac{az+b}{cz+d}$$

which maps the first quadrant $\{z : 0 < \text{Arg} z < \pi/2\}$ onto the region

$$\{z : \text{Im}\, z > 0, |z| > 1\},$$

and such that $f(1) = \infty$?

b) If the answer is positive, find this map, if negative, explain the reason.

Solution: a) Yes. These are digons with angles 90^o. Since the three points $0, 1, \infty$ on the boundary of the first digon are in the same order as the points $1, \infty, -1$ on the boundary of the second one, such a map exists. The triples contain corners of digons and the marked points 0 and ∞.

b) $f(z) = (1+z)/(1-z)$. Just use those two triples of points to find a, b, c, d.

7. Find all possible values of the integrals

$$\int_{\gamma} \frac{3z+1}{2(z^2-1)} dz,$$

for all closed curves γ which do not pass through $1, -1$.

Solution. The function under the integral has the following partial fraction decomposition

$$f(z) = \frac{1}{z-1} + \frac{1}{2(z+1)}.$$

So the integral equals $2\pi i(m + n/2)$, where m and n are arbitrary integers, and this can be written as $\pi i k$, where k is an arbitrary integer (positive, zero or negative).

8. How many roots does this equation have in the unit disk $\{z : |z| < 1\}$?

$$z^2 - 4z + e^z = 0.$$

Solution. For $|z| = 1$ we have

$$|z^2 + e^z| \leq 1 + e \approx 3.7 < 4 = |4z|,$$

so by Rouche's theorem, the number of roots is the same as of the larger function z, that is one root.