METRICS OF CONSTANT POSITIVE CURVATURE WITH FOUR CONIC SINGULARITIES ON THE SPHERE

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ABSTRACT. We show that for given four points on the sphere and prescribed angles at these points which are not multiples of 2π, the number of metrics of curvature 1 having conic singularities with these angles at these points is finite.

Key words: Heun's equation, accessory parameters, entire functions, surfaces, positive curvature.

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1. INTRODUCTION

We consider metrics of constant positive curvature with n conic singularities on the sphere. Without loss of generality we assume that curvature equals 1. Such a metric can be described by the length element $\rho(z)|dz|$, where z is a local conformal coordinate and ρ satisfies

$$\Delta \log \rho + \rho^2 = 2\pi \sum_{j=0}^{n-1} (\alpha_j - 1)\delta_{a_j},$$

where a_j are the singularities with angles $2\pi\alpha_j$. The problem is to describe the set of such metrics with prescribed singularities and angles.

For the recent results on the problem we refer to [6], [8], [9], [13], [14], [15]. It is believed that when none of the α_j is an integer, the number of such metrics with prescribed a_j and α_j is finite. This number has been found in some special cases, [23], [5], [7], [8], [9], [11], in particular, there is at most one such metric in the following two cases: a) when $n \leq 3$ and the angles are not multiples of 2π, [23], [5], and b) when $\alpha_j < 1$ for all j, and n is arbitrary, [11]. However for large angles and $n \geq 4$ there is usually more than one metric [3, 7, 8].

In this paper we address the case $n = 4$. We briefly recall the reduction of the problem to a problem about Heun's equation, see [7].

If S is the sphere equipped with such a metric, one can consider a developing map $f : S \to \overline{\mathbf{C}}$, where $\overline{\mathbf{C}}$ is the Riemann sphere equipped

with the standard metric of curvature 1. Strictly speaking, f is defined on the universal cover of $S\backslash\{a_0,\ldots,a_{n-1}\}$, but we prefer to consider f as a multi-valued function with branching at the singularities. One can write $f = w_1/w_2$ where w_1 and w_2 are two linearly independent solutions of the Heun equation, a Fuchsian equation with four singularities. These four singularities are the singularities of the metric, and the angles at the singularities are 2π times the exponent differences. Heun's equation can be written as

$$(1) \qquad w'' + \left(\sum_{j=0}^{2} \frac{1-\alpha_j}{z-a_j}\right) w' + \frac{Az-q}{(z-a_0)(z-a_1)(z-a_2)} w = 0,$$

where the singularities are a_0, a_1, a_2, ∞, the angles are $2\pi\alpha_j$, $0 \le j \le 3$, and

$$A = (2+\alpha_3-\alpha_0-\alpha_1-\alpha_2)(2-\alpha_3-\alpha_0-\alpha_1-\alpha_2)/4.$$

Three singularities can be placed at arbitrary points, so one can choose, for example $(a_0, a_1, a_2) = (0, 1, t)$. So for given singularities and angles, the set of Heun's equations essentially depends on 2 parameters: t which describes the quadruple of singularities up to conformal equivalence, and q which is called the *accessory parameter*.

The metric and the differential equation (1) can be lifted to a torus via the two-sheeted covering ramified over the four singular points. Assuming that the singularities are at e_1, e_2, e_3, ∞, where $e_1+e_2+e_3 = 0$, we consider the Weierstrass function

$$\wp : \mathbf{C} \to S,$$

with primitive periods ω_1, ω_2. We denote $\omega_0 = 0$, $\omega_3 = \omega_1 + \omega_2$, then $e_j = \wp(\omega_j/2)$, $1 \le j \le 3$. The resulting differential equation is called the Heun equation in the elliptic form:

$$(2) \qquad w'' = \left(\sum_{j=0}^{3} k_j\wp(z-\omega_j/2) + \lambda\right) w, \quad k_j = \alpha_j^2 - 1/4.$$

The two parameters are now the modulus of the torus $\tau = \omega_2/\omega_1$ and the accessory parameter λ. The relation between q and λ is affine: $\lambda = aq + b$ with a and b depending on τ. The precise form of a and b is irrelevant here, but we will use the fact that they are both real when a_j and α_j are real. The details of calculation reducing (1) to (2) are given in [21].

We will use both forms of the Heun equation. Equation (2) is considered on the torus (not on the plane). In particular the monodromy includes the translations of solutions by the elements of the lattice.

The ratio of two solutions $f = w_1/w_2$ is a developing map of a metric in question if and only if the projective monodromy group of the Heun equation is conjugate to a subgroup of $PSU(2) \simeq SO(3)$. In this case we say that the monodromy is *unitarizable*. The exponents at the singularity $\omega_j/2$ are $\rho_j^\pm = 1/2 \pm \sqrt{1/4 + k_j}$, so the angle at this singularity is $4\pi\sqrt{1/4 + k_j}$, which is $4\pi\alpha_j$, twice the angle of the original metric on the sphere.

The problem is for given ω_1, ω_2 and $k_j > -1/4$, $0 \le j \le 3$, to find the values of λ for which the monodromy is unitarizable.

The main result of this paper is

Theorem 1. *For every τ and $k_j > -1/4$, $0 \le j \le 3$, the set*

$U := \{\lambda \in \mathbf{C} : equation \ (2) \ has \ unitarizable \ projective \ monodromy\}$

is finite.

The correspondence between the metrics and Heun's equations is not one-to-one: different pairs of linearly independent solutions of the same equation may correspond to different metrics. This can only happen when the projective monodromy is *co-axial* that is isomorphic to a subgroup of the unit circle. We say that a metric is co-axial if the monodromy of its developing map is co-axial. Co-axial metrics come in continuous families consisting of *equivalent metrics*: two metrics with developing maps f_1, f_2 are called equivalent if $f_1 = \phi \circ f_2$ for some linear-fractional transformation ϕ. When the projective monodromy is trivial there is a real 3-parametric family of metrics, and when the projective monodromy is a non-trivial subgroup of the circle there is a real 1-parametric family of metrics, see, for example [4].

Co-axial metrics on the sphere have been completely described in [6]; in particular, when $n \ge 3$ some angles of a co-axial metric must be integer multiples of 2π. So Theorem 1 has the following

Corollary 1. *For every four points $a_0, \ldots, a_3$ on the Riemann sphere and every $\alpha_j \in \mathbf{R}_+ \backslash \mathbf{Z}$, there exist at most finitely many metrics of curvature 1 and conic singularities at a_j with angles $2\pi\alpha_j$.*

Whether a metric on the sphere is co-axial or not is completely determined by the angles [14, Theorem A]. The only co-axial metrics on surfaces of genus $g \ge 1$ are metrics on tori with all angles multiples of 2π [15, Theorem 2.1].

Corollary 2. *For every non-integer α, and every torus, there are at most finitely many metrics of curvature 1 and one conic singularity with angle α on this torus.*

Indeed, the developing map of such a metric is a ratio of two linearly independent solutions of (2) with $k_1 = k_2 = k_3 = 0$.

It follows from the results of Chen and Lin [3] that the number of metrics on a torus with one singularity with angle α is at least $[(\alpha - 1)/2] + 1$ unless α is an odd integer.

Unfortunately, our proof is non-constructive, and does not give any explicit upper estimate for the number of metrics. The proof of Theorem 1 consists of two parts: first we prove that the set U is bounded; this part is based on the asymptotic analysis of equation (2) as $\lambda \to \infty$.

Compactness of the set of metrics with prescribed angles in the given conformal class has been recently proved by Mondello and Panov [15] for metrics on arbitrary compact Riemann surfaces with any number of singularities, and for generic angles. For the case of 4 singularities on the sphere their condition on the angles is the following:

None of the sums $\sum_{j=0}^{3} \pm\alpha_j$ is an even non-zero integer.

The second part of our proof shows that the set of accessory parameters defining unitarizable monodromy is discrete. A general theorem of Luo [10] implies that equations with unitarizable monodromy correspond to a real analytic surface in the complex two dimensional space of all Heun's equations with prescribed exponents. Assigning the position of singularities means that we take the intersection of this real analytic surface with a complex line. In general, such an intersection does not have to be discrete.

Finding the accessory parameters corresponding to unitarizable monodromy requires solving a system of equations of the form

$$(3) \qquad\qquad g_j(\lambda) = 0, \quad j = 1, 2,$$

where λ is a complex variable and $g_j(\lambda)$ are real harmonic functions, and there is no general method of proving that the set of solutions of (3) is discrete, or to estimate the number of solutions from above. See [9], [2] where a very special case is solved.

To investigate equation (3) in our case, we use a general theorem of Stephenson [20] which reduces the local question about discreteness of the set of solutions of (3) to a question about asymptotic behavior at infinity of entire functions (traces of the generators of monodromy), and this question is solved using the asymptotic behavior established in the first part of the proof and a theorem of Baker [1] on compositions of entire functions.

It is the use of Stephenson's theorem that prevents our method from working for $n > 4$. In general, the set U as in Theorem 1 is defined as

a common zero set of $2n - 6$ real harmonic functions:

$$g_j(\lambda) = 0, \quad 1 \leq j \leq 2n - 6$$

of $n-3$ complex variables $\lambda = (\lambda_1, \ldots, \lambda_{n-3})$. When $n = 4$ we have two harmonic functions of one complex variable. If the set of common zeros is not discrete, it must be unbounded. This permits to use asymptotics as $\lambda \to \infty$ to obtain a contradiction. But when $n > 4$, the set of $2n - 6$ real harmonic equations in $n - 3$ variables may have a bounded non-discrete set of solutions, so our argument does not work.

Nevertheless we state the following

Conjecture. *On any compact Riemann surface, the set of Fuchsian equations with prescribed singularities and exponents, and with unitarizable monodromy is finite.*

The problem considered here is somewhat similar to the accessory parameter problem studied by Klein and Poincaré; see [17] for a modern exposition of their work. They tried to prove that there is a unique choice of accessory parameters such that the ratio of solutions of a Fuchsian differential equation is the inverse to the uniformizing map of S minus the punctures. They did not succeed in proving the Uniformzation theorem with this approach, but the proof based on these ideas is completed in [17]. It is interesting to notice that Poincaré did obtain a complete proof for the case of the sphere with four punctures [16]. This work of Poincaré was continued by V. I. Smirnov [18], [19] whose argument is used in Section 3 below. The main differences between our problem and the problem of Klein and Poincaré is that our problem may have more than one solution, and that the monodromy group in our case is not discrete.

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2. Asymptotics of traces of monodromy generators and boundedness of U

Let z_0 be a point such that no singularities of equation (2) lie on the lines

$$L_j = \{z_0 + t\omega_j : t \in \mathbf{R}\}, \ j = 1, 2.$$

We restrict equation (2) on the line L_j and obtain an equation with periodic analytic potential with period ω_j. Let $T_j(\lambda)$ be the monodromy transformation corresponding to the translation by ω_j. Then the well-known result [12], [22, Theorem 1] says that the trace $\operatorname{tr} T_j$ is an even

entire function of $\sqrt{\lambda}$ with the following asymptotic behavior:
(4)
$$\operatorname{tr} T_j(\lambda) = (2 + O(1/\lambda)) \cosh(\omega_j \sqrt{\lambda}), \quad \lambda \to \infty, \quad |\arg(\omega^2 \lambda)| < \pi - \epsilon,$$

for every $\epsilon > 0$. These traces are usually called Hill's discriminants or Lyapunov's functions in the literature on the Sturm-Liouville equations.

If the monodromy is unitarizable, both traces must satisfy

$$\operatorname{tr} T_j(\lambda) \in [-2, 2],$$

which is inconsistent with (4) for large λ since the ratio ω_2/ω_1 is not real. This proves that the set U in Theorem 1 is bounded.

3. The real case

To prove discreteness of the set U we first address the real case: we assume that $(a_0, a_1, a_2) = (0, 1, t)$, in (1) and that t and q are real. Here we essentially follow Smirnov [18], [19], who investigated the case of $SL(2, \mathbf{R})$ monodromy and $\alpha_j < 1$.

We only consider the case when none of the α_j in (1) is an integer. For the case of at least one integer α_j our result was proved in [8].

We assume without loss of generality that $t < 0$. Let w_{01}, w_{02} be solutions of (1) normalized at 0 by

$$w_{0j}(z) = z^{\rho_{0j}}(1 + g_{0j}(z)),$$

ρ_{0j} are the exponents at 0, $\rho_{00} = 0$, $\rho_{01} = \alpha_0$, and g_{0j} are holomorphic and vanish at 0. Here and in what follows we use the principal branches of powers. These two solutions w_{0j} are real on $(0, 1)$. We also consider two solutions w_{11}, w_{12} which are normalized at 1 and both are real on $(0, 1)$:

$$w_{1j}(z) = (1 - z)^{\rho_{1j}}(1 + g_{1j}(z)), \quad z \in (1 - \epsilon, 1),$$

where g_{1j} is analytic near 1, $g_{1j}(1) = 0$, and ρ_{1j} are the exponents at 1, $\rho_{11} = \alpha_1$, $\rho_{12} = 0$.

Then we have the connection matrix

$$F = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix},$$

such that

$$\mathbf{w}_0 = F\mathbf{w}_1, \quad \text{where} \quad \mathbf{w}_i = \begin{pmatrix} w_{i1} \\ w_{i2} \end{pmatrix}, \quad i \in \{0, 1\}.$$

The entries of F are entire functions of the accessory parameter q, real on the real line, see [18], [19]. To obtain the projective monodromy,

we consider $f_i = w_{i1}/w_{i2}$, $i \in \{0, 1\}$, which are related by a linear-fractional transformation $f_0 = L(f_1)$ which is represented by the matrix F. Projective monodromy of f_0 at 0 is an elliptic transformation with fixed points 0 and ∞. Since we assume that none of the α_j is an integer, the local monodromy at the singularities is not identical. Monodromy of f_0 at $z = 1$ is an elliptic transformation with fixed points $u_1 = f_{11}/f_{21}$ and $u_2 = f_{12}/f_{22}$. These points are real. Projective monodromies at 0 and 1 are simultaneously unitarizable if and only if the product of these fixed points in negative. Indeed, an elliptic transformation is a rotation of the Riemann sphere if and only if its fixed points u_1, u_2 are diametrally opposite, that is

$$(5) \qquad\qquad u_1 \overline{u_2} = -1.$$

In our case both u_1, u_2 are real so the bar can be dropped. Choosing the fixed points of projective monodromy at 0 to be $0, \infty$, we still can multiply f_0 by a constant μ. This will result in multiplying both fixed points of the projective monodromy at 1 by μ, so (5) can be achieved for these fixed points if and only if $u_1 u_2 < 0$.

Similar considerations apply to the interval $(t, 0)$. If we denote the connection matrix on $(t, 0)$ by

$$G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix},$$

then g_{ij} are entire functions on q, real on the real line, and the fixed points of the projective monodromy of f_0 at t are $v_1 = g_{11}/g_{21}$ and $v_2 = g_{12}/g_{22}$.

So the condition of unitarizability is

$$(6) \qquad\qquad \frac{f_{11}f_{12}}{f_{21}f_{22}} = \frac{g_{11}g_{12}}{g_{21}g_{22}} < 0.$$

This includes the condition that two meromorphic functions of q take equal values, so the set of q satisfying this condition is discrete, unless the equality in (6) is satisfied identically.

To show that the equation in (6) cannot be satisfied identically in q, one can use the asymptotics of solutions for large λ, but an easier way is to see this is directly from the Heun equation in the form (1), which in our case can be written as

$$w'' + p(z)w' + q(z)w = 0,$$

where

$$q(z) = \frac{Az - q}{z(z-1)(z-t)}, \quad t < 0.$$

When q is large negative, $q(z)$ is large negative on $(0,1)$, so solutions oscillate on $(0,1)$, and functions f_{ij} have infinitely many positive zeros, while on the interval $(t,0)$ we have $q(z) > 0$, solutions do not oscillate, and functions g_{ij} have no large positive zeros. Thus (6) cannot hold identically, and the set of real q for which the monodromy is unitarizable is discrete.

4. COMPLETION OF THE PROOF OF THEOREM 1

To prove the second part of Theorem 1, discreteness of the set U, we consider two entire functions $\lambda \mapsto \operatorname{tr} T_j(\lambda)$ introduced in section 2. They are the traces of the generators of the monodromy corresponding to the periods ω_1, ω_2.

If for some λ the monodromy is unitarizable, then $\operatorname{tr} T_j(\lambda) \in [-2, 2]$, so if the set of such λ is not discrete, there is a non-degenerate curve γ such that both $\operatorname{tr} T_j$ are real on γ.

Now we use the following

Theorem of Stephenson [18, Thm. 13]. *Let g_j, $j = 1, 2$ be two entire functions which are both real on a non-degenerate curve γ. Then*

$$(7) \qquad\qquad\qquad g_j = G_j \circ \phi,$$

where ϕ, G_j are entire, G_j are real on the real line, and ϕ is real on γ.

Recalling the asymptotics (4) we obtain

$$(8) \quad g_j(\lambda) := \operatorname{tr} T_j(\lambda) \sim 2\cosh(\sqrt{\omega_j^2 \lambda}), \quad \lambda \to \infty, \quad |\arg \omega^2 \lambda| \le \pi - \epsilon.$$

These two functions have two *different* directions of maximal growth and their zeros have arguments accumulating in the directions opposite to the directions of maximal growth. To be more precise, we say that an entire function g has a single direction of maximal growth θ if for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $r > r_0$ we have

$$\max\{|g(re^{it})| : \theta + \epsilon \le t \le \theta + 2\pi - \epsilon\} \le (1-\delta)\{\max |g(z)| : |z| = r\}.$$

Each of our functions g_j, $j = 1, 2$, has a single direction of maximal growth $\theta_j = \arg(\omega_j^{-2})$, and these directions are distinct because ω_1/ω_2 is not real. It follows that G_j in (7) cannot be polynomials, and ϕ cannot be a polynomial of degree greater than 1. Now we use

Theorem of Baker [1]. *If an entire function g of finite order has a representation (7) with some entire transcendental functions G and all zeros of g except finitely many lie in a sector of opening less than π, then ϕ must be a polynomial of degree 1.*

From this we conclude that the curve γ in Stephenson's theorem must be an interval of a straight line ℓ, and both g_j are symmetric with respect to this line, that is $g_j \circ s = \overline{g_j}$, where s is the reflection with respect to ℓ. Comparing this with asymptotics (8) we conclude that the directions of maximal growth of the g_j must be collinear, and since they are distinct, they must be opposite, which gives $\omega_1 = i\omega_2$. So our torus must be rectangular. In terms of equation (1), this means that the singularities are real. This implies that our functions g_j are real on the real line.

Now we prove that ℓ is the real line. First ℓ cannot cross the real line, because a function with two lines of symmetry will have at least two directions of maximal growth, while our functions have only one. Second, it cannot be parallel to the real line, because in this case our functions g_j would be periodic which is incompatible with their asymptotics (8).

This reduces the general case to the real case considered in the previous section and completes the proof of Theorem 1.

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