Harmonic analysis

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In this course we study boundary value problems for PDE. The main method is called *Harmonic analysis* a. k. a. *Fourier Analysis*. The main idea is that one can analyze a function by breaking it into simple parts.

Example. A function f defined on the real line is called *even* if $f(x) = f(-x)$ for all x. It is called *odd* if $f(x) = -f(-x)$ for all x. The only function which is simultaneously even and odd is the zero function.

For every function f, the function $f_e(x) = (f(x) + f(-x))/2$ is even, and the function $f_o(x) = (f(x) - f(-x))/2$ is odd. Since we have

$$f = f_0 + f_e,$$

every function can be represented as a sum of an even function and an odd function. Moreover, this representation is unique.

A function defined on the real line is called *periodic* with period $T \neq 0$ if

$$f(x+T) = f(x) \quad \text{for all} \quad x.$$

If f has period T then it also has periods nT, for every integer n. For a constant function, all numbers are periods. If f is not constant, continuous and periodic, then there exists the smallest positive period T and all other periods are nT, where n is any integer.

The simplest real periodic functions are cos and sin, but if we allow complex-valued functions, then there is even a simpler one

$$e^{ix} = \cos x + i \sin x,$$

whose smallest positive period is 2π. To obtain a similar function with period $2L$ we take

$$e^{ix/L} = \cos(\pi x/L) + i \sin(\pi x/L).$$

What do we mean by "simplest" here? This has a precise answer: these functions behave in a very simple way when we differentiate them, especially the exponential: $(d/dx)e^{\lambda x} = \lambda e^x$, in other words the exponential is an *eigenfunction* of the differentiation operator.

The simplest setting of Harmonic Analysis is the theory of *Fourier series* which gives an expansion of a *periodic function* f with period $2L$ into a Fourier series of the form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{i\pi n x}{L}\right),$$

where c_n are complex constants. We will frequently take $L = \pi$ to simplify our formulas; it must be clear how to modify them for arbitrary period.

So suppose that we have a 2π-periodic function which has an expansion of the form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}. \tag{1}$$

How can one determine these constants c_n (they are called *Fourier coefficients*) for given f? There is a simple recipe: *multiply both sides on* e^{-imx} *and integrate over the period* 2π. Assuming that the series can be integrated term-by-term, we use

$$\int_{-\pi}^{\pi} e^{inx} e^{-imx} dx = \int_{-\pi}^{\pi} e^{i(n-m)x} = \begin{cases} 0, & m \neq n \\ 2\pi, & m = n. \end{cases}$$

Using these relations, we obtain

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx, \quad -\infty < n < \infty. \tag{2}$$

These formulas (1), (2) define the special case of *Fourier transform*: to a function with period 2π we put into correspondence a two-sided sequence of complex numbers (c_n) by formula (2) and (presumably) the function can be recovered from this sequence by formula (1).

Of course, all this needs a mathematical justification: why integral in (2) exists? If it exists, why the series (1) is convergent, and in what exact sense? If it is convergent, why its sum is f? And why we can integrate it term-by-term to obtain (2)? All this has to be justified, and a precise class of functions has to be defined for which all these arguments work.

Since Fourier's original discoveries, these questions always occupied mathematicians, and research on these questions continues. The modern definitions of the basic mathematical notions, like "function" and "integral" and "convergence" actually developed in the process of this research.

Some basic facts are the following. *Suppose that f is 2π-periodic and* **piecewise smooth**. *Then the Fourier series of f exists and converges at every point x to the value $(f(x-0)+f(x+0))/2$.*

In particular, at all points x where f is continuous, the Fourier series converges to $f(x)$.

Moreover, *once we have an expansion (1), the numbers c_n must be defined by formulas (2);* in other words, Fourier expansion is unique.

For the proof and precise definition of "piecewise smooth", see Section 2.2 of the book. More advanced theorems on Fourier correspondence will be discussed later.

All of the above applies to functions of the real variable which can take complex values. Let us see what happens if the function f is real (takes only real values). This case is of course important in applications. The condition that f is real, can be stated as $\overline{f} = f$. Applying complex conjugation to all terms of (1) we obtain

$$\overline{f(x)} = \sum_{n=-\infty}^{\infty} \overline{c_n} e^{-inx}.$$

Since the LHS's of this formula and (1) are equal, the RHS's must be also equal, and this implies that $\overline{c_n} = c_{-n}$ for all n, since *the Fourier coefficients are uniquely determined by f.*

Now let $c_n = A_n + iB_n$, where A_n and B_n are real, then the condition $\overline{c_n} = c_{-n}$ means that $A_n = A_{-n}$ and $B_{-n} = -B_n$. Writing the general term of the Fourier series as

$$(A_n+iB_n))(\cos nx+i\sin nx) = A_n\cos nx - B_n \sin nx + i(A_n \sin nx + B_n \cos nx),$$

we can group each summand with n with the one with $-n$, then all imaginary terms in (1) will cancel (as it should be!) and we obtain

$$f(x) = A_0 + \sum_{n=1}^{\infty} \left(2A_n \cos nx - 2B_n \sin nx\right).$$

For (2) we will have

$$A_n + iB_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(\cos nx - i \sin nx)dx,$$

so, since f is assumed real,

$$A_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \quad B_n = -\frac{1}{2\pi}\int f(x)\sin nx dx.$$

It is convenient to define $a_n = 2A_n$ and $b_n = -2B_n$, and with these notation we obtain *Fourier formulas for real functions*:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx, \tag{3}$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx. \tag{4}$$

These were the original formulas of Fourier. The reason of writing the constant term as $a_0/2$ is that we have a unified formula (4) for a_n for all n, including $n = 0$. Notice that for $n = 0$ we have

$$\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx,$$

the *average* of f over the period.

Notice that the first two summands in the RHS of (3) are even while the last term is odd. So we obtained a decomposition of a real function into the even and odd part which was addressed in the beginning of this text, in terms of Fourier series:

$$f_e(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx,$$

$$f_o(x) = \sum_{n=1}^{\infty} b_n \sin nx,$$

where a_n, b_n are defined in (4)

Example. Suppose that

$$x + 1 = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right), \quad -\pi < x < \pi$$

. What is the sum of the series in the RHS for $x = \pi$?

Solution. To obtain an expansion on the whole real line, we need to extend f periodically, with period 2π. We do not know the values of this extension at the odd multiples of π. But they are irrelevant in Fourier formulas (4), so the right hand side does not depend of them. No matter how $f(\pi)$ and $f(-\pi)$ are defined, the function is piecewise smooth, and the Convergence theorem stated above must hold. Assording to this theorem, the RHS converges at the point π to the value $(\tilde{f}(\pi-0)+\tilde{f}(\pi+0))/2$ where $\tilde{f}$ is the 2π-periodic extension of f. This value is equal to $((\pi+1)+(-\pi+1))/2=1$.

Example. Suppose that we have the Fourier expansion

$$xe^x = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right), -\pi < x < \pi.$$

Find a_0, and

$$S(x) := \sum_{n=1}^{\infty} b_n \sin nx, \quad -\pi < x < \pi.$$

Solution. a_0 is twice the average:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} xe^x dx = (\pi-1)e^{\pi} + (\pi+1)e^{-\pi}.$$

And S is the odd part of our function:

$$S(x) = (xe^x - (-x)e^{-x})/2 = x\cosh x.$$

Of course one could compute all coefficients b_n by formulas (4) and then find the sum of the series, but this is much more difficult.