Examples of partial fraction decomposition

Example 1

$$\frac{2x^2+2x+13}{(x-2)(x^2+1)^2} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}.$$

Adding the fractions in the RHS, we write that the numerators are equal:

$$2x^2+2x+13 = A(x^2+1)^2 + (Bx+C)(x^2+1)(x-2) + (Dx+E)(x-2).$$

Writing that the coefficients at equal degrees are equal, we obtain the system of equations:

$$\begin{array}{c|rl} x^4 & A+B & = 0, \\ x^3 & -2B+C & = 0, \\ x^2 & 2A+B-2C+D & = 2, \\ x^1 & -2B+C-2D+E & = 2, \\ x^0 & A-2C-2E & = 13, \end{array}$$

whose solution is

$$A=1, \quad B=-1, \quad C=-2, \quad D=-3, \quad E=-4.$$

Thus

$$\frac{2x^2+2x+13}{(x-2)(x^2+1)^2} = \frac{1}{x-2} - \frac{x+2}{x^2+1} - \frac{3x+4}{(x^2+1)^2}.$$

Example 2

$$\frac{1}{x^2(1+x^2)^2} = \frac{1}{x^2} - \frac{1}{1+x^2} - \frac{1}{(1+x^2)^2}.$$

Example 3

$$\frac{4x^2+4x-11}{(2x-1)(2x+3)(2x-5)} = \frac{A}{x-1/2} + \frac{B}{x+3/2} + \frac{C}{x-5/2}.$$

Notice that the denominators of fractions should be of the form $(x-a)$, not of the form $ax+b$. When all linear factors in the denominator are *different* (so there are no powers), one can apply a shortcut. For example, to find A we multiply our equality by $x-1/2$ and put $x=1/2$. Everything in the RHS, except A will disappear, and we obtain $A=1/4$ just by plugging $1/2$ to

$$\frac{4x^2+4x-11}{2(2x+3)(2x-5)}.$$

Similarly, to obtain B we multiply by $x+3/2$ and plug $x=-3/2$, which gives $B=-1/8$. Similarly we obtain $C=3/8$.

Example 4, using complex numbers.

$$\frac{1}{x^2+1}=\frac{A}{x+i}+\frac{B}{x-i}.$$

Using the same argument as in Example 3, we obtain $A=1/(-i-i)=i/2$ and $B=-i/2$.