Fourier transform

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1. Definition and heuristic arguments. Let us write Fourier expansion of a function on an interval $(-L, L)$ in the form

$$c_n = \int_{-L}^{L} f(x)e^{-i\pi nx/L}dx, \quad -\infty < n < \infty, \tag{1}$$

$$f(x) = \frac{1}{2L}\sum_{-\infty}^{\infty} c_n e^{i\pi nx/L}, \quad -L < x < L. \tag{2}$$

In comparison with usual formulas, we shifted the multiple $1/(2L)$ from the first formula to the second.

Denote $s_n = \pi n/L$, $\Delta s = s_{n+1} - s_n = \pi/L$, and introduce a function $F(s_n) = c_n$; then $1/(2L) = \Delta s/(2\pi)$, and our formulas become

$$F(s_n) = \int_{-L}^{L} f(x)e^{-is_nx}dx,$$

$$f(x) = \frac{1}{2\pi}\sum_{-\infty}^{\infty} F(s_n)e^{is_nx}\Delta s.$$

We can use the first formula to define F for all values of s, not only for s_n, then the second formula becomes an integral sum for some integral. Then we let $L \to \infty$ in which case $\Delta s \to 0$, and the integral sums may be expected to converge to the integral. This is a heuristic justification of the following definition:

$$F(s) = \int_{-\infty}^{\infty} f(x)e^{-isx}dx. \tag{3}$$

This is called the *Fourier transform* of f. And we expect the following *inversion formula*

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) e^{isx} ds. \tag{4}$$

Now we have to discuss the conditions for which these formulas make sense and are true.

First of all, the integral in (3) must be convergent. A sufficient condition for this is $f \in L^1$, that is

$$\|f\|_1 := \int_{-\infty}^{\infty} |f(x)| dx < \infty.$$

Second, we need the integral (4) to be convergent (it is not true that $F \in L^1$ for every $f \in L^1$, as seen from the examples below), and if it makes sense, we have to prove that it is really equal to f.

Before discussing this, we establish some formal properties, assuming that both Fourier transform and the inverse transform are well defined.

We will use the following notation: if F is a Fourier transform of f, we will denote it by $F = \hat{f}$, or by $F[f]$, whichever is more convenient. Notice that a function and its Fourier transform are functions of *different variables*, so these variables have to be denoted by different letters when they occur in the same formula. Fourier transform and the inverse transform are very similar, so to each property of Fourier transform corresponds the *dual property* of the inverse transform.

2. Properties of Fourier transform.

1. Fourier transform is linear:

$$F[af + bg] = aF[f] + bF[g].$$

2. Fourier transform of a shifted function:

$$F[f(x-a)] = e^{-ias}\hat{f}(s), \quad \text{and} \quad F[e^{iax}f(x)] = \hat{f}(s-a).$$

3. For a function f and a real number $\delta \geq 0$, we define $f_\delta(x) = \delta^{-1} f(x/\delta)$. A justification of this definition is that

$$\int_{-\infty}^{\infty} f_\delta(x) dx = \int_{-\infty}^{\infty} f(x) dx,$$

by the change of the variable $x/\delta = y$. So this is a scaling which does not change the integral. Then

$$F[f_\delta] = \hat{f}(\delta s), \quad \text{and} \quad F[f(\delta x)] = \hat{f}_\delta.$$

4. If f is differentiable and $f' \in L^1$, then

$$F[f'] = is\hat{f}(s),$$

and if $xf(x) \in L^1$ then

$$F[xf(x)] = i\hat{f}'(s).$$

The first formula is proved by integration by parts, and the second by differentiation under the integral sign.

Applications to differential equations are based on this property 4: Fourier transform transforms differentiation to multiplication on the independent variable.

The following examples are very important and will be used later:

Example 1. Let $\chi_a(x) = 1/(2a)$ when $|x| \leq a$ and $\chi_a(x) = 0$ otherwise, Then

$$F[\chi_a](s) = \frac{1}{2a}\int_{-a}^{a} e^{-isx}dx = \frac{\sin(as)}{as}, \quad a > 0.$$

Remark. Notation χ_a is consistent with the scaling introduced above: $\chi_a(x) = (1/a)\chi_1(x/a)$.

Example 2.

$$F[e^{-ax^2/2}] = \sqrt{\frac{2\pi}{a}}e^{-s^2/(2a)}, \quad a > 0. \tag{5}$$

See the handout "Some definite integrals" where this is discussed.

Here is another approach to this formula, based on property 4 of Fourier transform. Let $y(x) = \exp(-ax^2/2)$. Then

$$y' = -axy.$$

Taking Fourier transform of both sides, and using property 4, we obtain

$$is\hat{y} = -ia\hat{y}', \quad \text{that is} \quad \hat{y}' = -(s/a)\hat{y}.$$

Solving the last differential equation, we obtain

$$\hat{y} = C(a)e^{-s^2/(2a)}.$$

To determine the constant $C(a)$, we plug $s = 0$ to the formula

$$\hat{y}(s) = \int_{-\infty}^{\infty} e^{-ax^2/2} e^{-isx} dx = C(a) e^{-s^2/(2a)}$$

and use the result that

$$\int_{-\infty}^{\infty} e^{-ax^2/2} dx = \sqrt{\frac{2\pi}{a}},$$

proved in the handout "Some definite integrals".

Putting $a = 1$ in (5) we obtain that

$$f(x) = e^{-x^2/2} \quad \text{has Fourier transform} \quad \sqrt{2\pi} e^{-s^2/2}, \tag{6}$$

in other words, it is an eigenfunction of the Fourier operator F with eigenvalue $\sqrt{2\pi}$.

Formula (6) is very important, and it is recommended for memorizing. The general formula (5) can be obtained from it by the scaling rule (property 3 in the list of the properties of Fourier transform with $\delta = 1/\sqrt{a}$).

Example 3.

$$F[e^{-a|x|}] = \frac{2a}{s^2 + a^2}, \quad F[(x^2 + a^2)^{-1}] = \frac{\pi}{a} e^{-a|s|}.$$

This is left as an exercise: for the first formula, just break the integral into two parts. The second formula follows by duality.

Exercise. Prove the formulas

$$\hat{\hat{f}}(x) = 2\pi f(-x), \quad \text{and} \quad \hat{\bar{f}}(s) = \bar{\hat{f}}(-s).$$

The next important property of Fourier transform is related to

3. Convolution.

Let f and g be two functions on the real line. We define their convolution $f \star g$ as the function

$$(f \star g)(x) = \int_{-\infty}^{\infty} f(x - y) g(y) dy,$$

assuming that the integral is absolutely convergent. For example, this will be the case if one function is in L^1 while the other is bounded. Or if both functions are in L^2.

Exercise. Prove the last statement by using the Cauchy–Schwarz inequality.

To understand what this operation really does, consider the example in which $g(y) = \chi_a(y) = 1/(2a)$, when $|x| \leq a$ and 0 otherwise. Then

$$(f \star \chi_a)(x) = \frac{1}{2a} \int_{-a}^{a} f(x-y)dy = \frac{1}{2a} \int_{x-a}^{x+a} f(y)dy,$$

so $f \star \chi_a$ is the "moving average": its value at the point x is the average of f over the interval of length $2a$ centered at x. Similar interpretation has a convolution with any positive function $g \in L^1$: it is a constant times are moving weighted average with weight g.

We state the main properties of convolution.

1. Convolution is commutative: $f \star g = g \star f$. Indeed,

$$(f \star g)(x) = \int f(x-y)g(y)dy = \int g(x-t)f(t)dt = (g \star f)(x),$$

where we made the change of the variable $y = x - t$.

2. Convolution is associative:

$$(f \star g) \star h = f \star (g \star h).$$

This is also proved by a change of the variable in the integral.

3. Convolution is linear with respect to each argument:

$$f \star (ag_1 + bg_2) = af \star g_1 + bf \star g_2,$$

and similarly with respect to f.

4. If one of the functions is continuous and has bounded support[1], then the convolution is defined and is is continuous; if one of the functions is n times continuously differentiable and has bounded support, then the convolution is

[1] Support of a function is the set where it is different from 0.

n times continuously differentiable. Roughly speaking convolution is at least as good as the better of the two functions. This is proved by differentiation under the integral sign. The requirement of bounded support can be relaxed by assuming sufficiently fast decrease at infinity, so that all integrals are convergent.

Property 4 can be used to approximate arbitrary functions by smooth functions.

Example. Show that there is an infinitely differentiable function which is positive on the positive ray and zero on the negative ray. For example,

$$f(x) = \begin{cases} e^{-1/x}, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

One only has to check that it is infinitely differentiable at 0, and all derivatives "from the right" at 0 are equal to 0. For $x > 0$, we have

$$f'(x) = x^{-2}e^{-1/x}, \quad f''(x) = (-2x^{-3} + x^{-4})e^{-1/x},$$

and so on. It is easy to see that $f^{(n)}$ is of the form: a rational function times $\exp(-1/x)$. All these functions tend to 0 as $x \to 0$.

Now the function $g(x) = cf(x)f(1 - x)$ is non-negative, equals to zero outside of the interval $(-1, 1)$ and is infinitely differentiable; and we choose the constant c in such a way that integral of g over the whole real line equals 1. Then also

$$\int g_\delta(x)dx = \int g(x)dx = 1 \quad \text{for all} \quad \delta > 0,$$

where $g_\delta(x) = \delta^{-1}g(x/\delta)$.

Now, if f is any continuous function, then $f \star g_\delta$ is well defined, infinitely differentiable, and tends to f as $\delta \to 0$ pointwise.

Exercise. Prove the last statement.

Now we state one of the main properties of the Fourier transform:

Theorem. *Fourier transform of a convolution is the product of Fourier transforms:*

$$F[f \star g] = \hat{f}\hat{g}.$$

And we have the dual property:

$$F[fg] = \frac{1}{2\pi}\hat{f} \star \hat{g}.$$

Proof.

$$\begin{aligned} F[f \star g] &= \int e^{-isx} \int f(x-y)g(y)dydx \\ &= \int f(x-y)e^{-is(x-y)}dxg(y)e^{-isy}dy = \hat{f}(s)\hat{g}(s), \end{aligned}$$

where we used the fact that shift of a function does not change its integral over the real line. The proof of the dual property is similar.

Now we briefly discuss two approaches to a rigorous definition of Fourier transform and the proof of the inversion formula.

4. Schwartz's space.

By definition, it consists of infinitely differentiable functions which decrease sufficiently fast at infinity, so that

$$\sup_{\mathbf{R}} f^{(n)}(x)(1+|x|^m) < \infty \quad \text{for all positive integers} \quad m, n.$$

Evidently, such functions make a vector space. Examples of such functions are infinitely differentiable functions with bounded support which we constructed earlier. Another important example is e^{-ax^2}, $a > 0$.

The set of all such functions is called the *Schwartz space* and denoted by $\mathscr{S}$. (This is a fancy Latin letter "S", in honor of Laurent Schwartz). This definition is inspired by property 4 of Fourier transform: you can multiply $f \in \mathscr{S}$ on x^m and differentiate any number of times, the Fourier transform will be differentiated and multiplied by powers of s, and the integrals defining Fourier transform will be always convergent. The formal properties are:

Fourier transform maps S into itself, and this map is one-to-one.

Fourier Inversion formula holds for functions of class $\mathscr{S}$.

Proof. First we verify the identity

$$\int_{\infty}^{\infty} f(x)\hat{g}(x)dx = \int_{-\infty}^{\infty} \hat{f}(x)g(x)dx, \tag{7}$$

which holds for any two functions of class $\mathscr{S}$. To prove the (7), it is sufficient to notice that both sides are equal to the double integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)e^{-ixy}g(y)dxdy.$$

Now we apply this identity to a function $f \in \mathscr{S}$ with

$$g(\delta, x) = \frac{1}{2\pi} e^{-\delta x^2},$$

whose Fourier transform is

$$F[g(\delta, x)] = \frac{1}{\delta\sqrt{2\pi}} e^{-x^2/2\delta^2},$$

according to Example 2 with $a = 2\delta$. As $\delta \to 0$, the LHS of (7) tends to $f(0)$ while in the RHS $e^{-\delta x^2} \to 1$, so

$$f(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(x) dx.$$

This established the inversion formula for $x = 0$.

Applying property 2 of Fourier transform we obtain the inversion formula in full generality. Since both Fourier transform and the inversion formula are defined for all functions of $\mathscr{S}$, Fourier correspondence on $\mathscr{S}$ is one-to-one.

So class $\mathscr{S}$ is very convenient, for Fourier transform. Its disadvantage is that it contains too few functions. Many important functions are not infinitely smooth, but only piecewise smooth (for example the function χ_a that we considered above), other functions do not tend to zero sufficiently fast (like Example 3 above: f is not continuously differentiable, while $\hat{f}$ does not decrease sufficiently fast.)

5. Space L^2.

It consists of all functions on the real line with the property

$$\|f\|_2^2 := \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty,$$

and the dot product is defined by the formula

$$(f, g) = \int_{-\infty}^{\infty} f(x)\overline{g(x)} dx,$$

the last integral is convergent by the Cauchy–Schwarz inequality. It turns our that Fourier transform just multiplies the dot product by a constant. To verify this, we use the formula which follows immediately from the definition of Fourier transform:

$$\hat{\overline{g}}(s) = \overline{\hat{g}}(-s).$$

Now

$$2\pi(f,g) = 2\pi F[f\overline{g}](0) = (\hat{f} \star \hat{\overline{g}})(0) = (\hat{f}\bar{\hat{g}}(-s))(0) = (\hat{f},\hat{g}).$$

In particular,

$$2\pi\|f\|_2^2 = \|\hat{f}\|_2^2. \tag{8}$$

This is a Fourier Transform analog of the Parseval formula in the theory of Fourier series. In the context of Fourier transform on the real line, (8) is called the Plancherel formula.

So once Fourier transform and the inversion formula is established for some nice functions, like class $\mathscr{S}$ functions, one can extend it to all L^2-limits. One can show that functions of class $\mathscr{S}$ are dense in L^2, that is every function in L^2 can be approximated in the sense of L^2 distance by functions of class $\mathscr{S}$. This shows that

Fourier transform is a one-to-one map of L^2 onto itself. It multiplies the norm of every function by $\sqrt{2\pi}$.

Plancherel's formula shows that all distances are scaled by $\sqrt{2\pi}$ under Fourier transform, and all angles are preserved.

These two methods of defining Fourier transform, in class $\mathscr{S}$ or in L^2 still do not have all desired properties. For example, none of these classes contains periodic functions. So some further extension is needed. An ingenuous way to do this was invented by Laurent Schwartz (not to be confused with Hermann Amandus Schwarz of the Cauchy–Schwarz inequality!). We will address this is one of the future lectures.

6. Real functions. One-sided real Fourier transforms.

In many applications, function f is real, and it is sometimes useful to rewrite our formulas so that they involve real functions only.

Suppose that f is even. Then we can break the Fourier integral into two parts:

$$\begin{aligned}\hat{f}(s) &= \int_{-\infty}^{\infty} f(x)e^{-isx}dx \\ &= \int_0^{\infty} f(x)\left(e^{-isx}+e^{isx}\right)dx = 2\int_0^{\infty} f(x)\cos(sx)dx.\end{aligned}$$

The last integral is called the cosine Fourier transform. Notice that it is also even since cosine is even.

Similarly, if f is odd,

$$\hat{f}(s) = -2i \int_0^\infty f(x)\sin(sx)dx =: -iS(s),$$

where S is called the sine Fourier transform. Let us denote the cosine transform by $C[f]$ and sine transform by $S[f]$. If f is real, they are also real.

Then we can use the inversion formula. In the case of even functions

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(s)e^{isx}ds \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} C[f]e^{isx}ds = \frac{1}{\pi}\int_0^\infty C[f](s)\cos(sx)ds. \end{aligned}$$

and similarly for $S[f]$:

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} S[f](s)\sin(sx)ds.$$

So we have real formulas which can be used for functions defined on the positive ray, or for even (and odd) functions defined on the whole real line.

7. Poisson's Summation Formula.

Suppose we have a function f of class $\mathscr{S}$. I recall that this means that it is infinitely differentiable, and all derivatives tend to 0 faster than any power as $x \to \pm\infty$.

One can make a periodic function with period 1 out of f in the following way

$$f_1(x) = \sum_{-\infty}^{\infty} f(x+n),$$

this is sometimes called the *periodization* of f. The series is convergent since f tends to zero faster than any power.

There is another way to produce a periodic function from f: in the inverse Fourier transform

$$f(2\pi x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(s)e^{2\pi isx}ds$$

replace the integral by the sum over the arithmetic progression $2\pi n$ and multiply by 2π:

$$f_2(x) := \sum_{-\infty}^{\infty} \hat{f}(2\pi n) e^{2\pi i n x}.$$

The series is convergent because $\hat{f} \in \mathscr{S}$, and evidently f_2 is periodic with period 1. It turns out that these two methods of periodization give the same result: $f_1 = f_2$.

Theorem (Poisson's summation formula). For $f \in \mathscr{S}$,

$$\sum_{-\infty}^{\infty} f(x+n) = \sum_{-\infty}^{\infty} \hat{f}(2\pi n) e^{2\pi i n x}.$$

Proof. Both sides are periodic with period 1. So to check the equality, it is sufficient to check that Fourier coefficients of both sides coincide. The RHS is already a Fourier series, so Fourier coefficients are $\hat{f}(2\pi n)$. For the LHS we have

$$\begin{aligned}\int_0^1 \left(\sum_{-\infty}^{\infty} f(x+n) \right) e^{-2\pi i n x} dx &= \sum_{-\infty}^{\infty} \int_0^1 f(x+n) e^{-2\pi i n x} dx \\ = \sum_{-\infty}^{\infty} \int_n^{n+1} f(y) e^{-2\pi i n y} dy &= \int_{\infty}^{\infty} f(y) e^{-2\pi i n y} dy = \hat{f}(2\pi n).\end{aligned}$$

Putting $x = 0$ we obtain the identity

$$\sum_{-\infty}^{\infty} f(n) = \sum_{-\infty}^{\infty} \hat{f}(2\pi n).$$

Taking $f(x) = \exp(-ax^2/2)$ (Example 2 above), we obtain

$$\sum_{-\infty}^{\infty} e^{-an^2/2} = \sqrt{\frac{2\pi}{a}} \sum_{-\infty}^{\infty} e^{-2\pi^2 n^2/a}, \quad a > 0.$$

This is useful when we want to compute the LHS approximately when a is very small: when $a \to 0+$, the series in the LHS becomes divergent. But the terms of the series in the RHS become very small. So LHS$\sim \sqrt{2\pi/a}, \quad a \to 0+$. See " Age of the Earth, 2" where this is used.

Exercise. Derive a more general formula:

$$\sum_{-\infty}^{\infty} f(x+nT) = \frac{1}{T}\sum_{-\infty}^{\infty} \hat{f}(2\pi n/T)e^{2\pi i n x/T}.$$

Heisenberg's inequality.

Let $f \in \mathscr{S}$, be a function such that $\|f\|^2 = 1$. Then

$$\int_{-\infty}^{\infty} x^2|f(x)|^2dx \int_{-\infty}^{\infty} s^2|\hat{f}(s)|^2ds \geq \frac{\pi}{2}.$$

Proof. By integration by parts we obtain

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} |f(x)|^2dx = -\int_{-\infty}^{\infty} x\frac{d}{dx}|f(x)|^2dx \\ &= -\int_{-\infty}^{\infty} \left(xf'(x)\overline{f(x)} + xf(x)\overline{f'(x)}\right)dx, \end{aligned}$$

where we used $|f|^2 = f\overline{f}$. So

$$1 \leq 2\int_{-\infty}^{\infty} |x||f(x)||f'(x)|dx.$$

Applying Cauchy-Schwarz inequality we obtain

$$1 \leq 4\|xf(x)\|^2\|f'\|^2.$$

On the other hand, by property 4 of Fourier transform and Plancherel's formula,

$$\|f'\|^2 = \frac{1}{2\pi}\|F[f']\|^2 = \frac{1}{2\pi}\|s\hat{f}(x)\|^2,$$

and we obtain our inequality.

Exercise 1. Take $f(x) = e^{-xs_0}g(x+x_0)$, where $g \in \mathscr{S}$, $\|g\| = 1$, and obtain a more general inequality:

$$\int_{-\infty}^{\infty} (x-x_0)^2|g(x)|^2dx \int_{-\infty}^{\infty} (s-s_0)^2|\hat{g}(s)|^2ds \geq \frac{\pi}{2},$$

which holds for all $g \in \mathscr{S}$ with $\|g\| = 1$ and all real s_0, x_0.

Exercise 2. Find when equality holds in Heisenberg's inequality. Hint: when does equality hold in Cauchy–Schwarz inequality?