Rational functions with real critical points and the B. and M. Shapiro conjecture in real enumerative geometry

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December 15, 1999

Abstract

Suppose that $2d-2$ tangent lines to the rational normal curve of degree d, $z \mapsto (1 : z : \ldots : z^d)$ in d-dimensional projective space are given. It was known that the number of codimension 2 subspaces intersecting all these lines is always finite; for a generic configuration it is equal to the d-th Catalan number. We prove that for real tangent lines, all these codimension 2 subspaces are also real. This confirms a special case of a general conjecture of B. and M. Shapiro, which has applications to the problem of pole assignment in the theory of automatic control. Our result is equivalent to the following:

If all critical points of a rational function lie on a circle in the Riemann sphere (for example, on the real line), then the function maps this circle into a circle.

1. Introduction

We may assume that the first circle is the real line $\mathbb{R} \cup \infty$.

Two rational functions f_1 and f_2 will be called equivalent if $f_1 = \ell \circ f_2$, where ℓ is a fractional-linear transformation. Equivalent rational functions have the same critical sets.

*Supported by NSF grant DMS-9800084 and by Bar Ilan University.
†Supported by NSF grant DMS-9704745

Theorem 1 *If all critical points of a rational function are real then it is equivalent to a real rational function.*

Lisa Goldberg [9] addressed the following question: how many equivalence classes of rational functions of degree d with a given critical set of $2d-2$ points may exist? She reduced this to the following problem of enumerative geometry:

Problem P. *Given* $2d-2$ *lines in general position in projective space* $\mathbb{CP}^d$*, how many projective subspaces of codimension* 2 *intersect all of them?*

The answer to this question, going back to Schubert (e. g. [11]), is

$$u_d = \frac{1}{d}\begin{pmatrix} 2d-2 \\ d-1 \end{pmatrix}, \quad \text{the } d\text{-th Catalan number.} \tag{1}$$

So the result is

Theorem A (L. Goldberg [9]). *The number of equivalence classes of rational functions of degree d with given* $2d-2$ *distinct critical points is at most* u_d*.*

We prove

Theorem 2 *For given* $2d-2$ *distinct real points, there exist at least* u_d *classes of real rational functions of degree d with these critical points.*

Theorems A and 2 imply Theorem 1. In general, even if the lines in Problem P are real, the subspaces of codimension two might not be real [11]. Fulton [7] asked the following general question: how many solutions of a problem of enumerative geometry can be real, when that problem is one of counting geometric figures of some kind having specified position with respect to some general fixed figures. A specific conjecture for the Problem P was made by Boris and Michael Shapiro [17]: if the lines in question are tangent to the rational normal curve at $2d-2$ real points, then all u_d solutions of the problem are real. Our Theorem 2 implies that this conjecture is true. Notice that every irreducible nondegenerate rational curve is equivalent to the rational normal curve via a projective transformation [10, p. 299]

Another way to reformulate our result is the following. A non-constant rational function of degree at most d is a ratio of two non-proportional polynomials of degree at most d. This leads to parametrization of classes of rational functions by points of the Grassmanian $G(2,d)$. Critical sets of these functions can be parametrized by points of $\mathbb{CP}^{2d-2}$ (precise definition of these parametrizations is given in Section 7). We have a regular map $\widetilde{W} : G(2, d+1) \to \mathbb{CP}^{2d-2}$, defined by taking the Wronskian determinant of a pair of polynomials. Theorem A says that this map is finite and has degree u_d. Our Theorem 1 implies that $\widetilde{W}$ is unramified over the real part $\mathbb{RP}^{2d-2}$.

For a general discussion of the B. and M. Shapiro conjectures, with ample experimental evidence, bibliography and connections with automatic control theory, we refer to the web site [16]. For the closely related problem of pole assignment in the theory of automatic control we refer to [5, 6].

As a corollary from his main result in [15], Sottile proved that there exists an open (in the usual topology) set $X \subset \mathbb{R}^{2d-2}$, such that for $x \in X$ there exist u_d classes real rational functions of degree d, whose sequence of critical points is x. Theorem 2 was also proved by Sottile for $d = 3$, and verified, using computers, for $d \leq 9$. The computation for $d = 9$ ($u_9 = 1,430$) is due to Verschelde [18].

It is interesting that our proof of Theorem 1 is based on the fact that two different combinatorial problems have the same sequence of integers as their solution. These two combinatorial problems are Problem P and the one in Lemma 1 below. We prove Theorem 2 in Sections 2–6 and derive Theorem 1 in Section 7.

The scheme of our proof of Theorem 2 is following. We consider the unit circle $\mathbb{T}$ instead of the real line. Let f be a rational function of degree d, mapping $\mathbb{T}$ into itself, having $2d-2$ distinct critical points in $\mathbb{T}$, and properly normalized. We introduce a "net" $\gamma(f) = f^{-1}(\mathbb{T})$, considered modulo symmetric normalized homeomorphisms of the Riemann sphere. Classes of nets are combinatorial objects, describing topological properties of rational functions f. To describe a function f of our class completely, we need one more piece of data, which we call labeling. It is a function on the set of edges of a net, which assigns to each edge the spherical length of its image. We give a precise description of all nets γ and labelings, which may occur. It is important that, for a fixed γ, the space of possible labelings has simple topological structure: it is a convex polytope. This leads to a parametrization of a set R of normalized rational functions mapping $\mathbb{T}$ into itself, with all critical points in $\mathbb{T}$, by equivalence classes of labeled nets. Similar parametrizations for polynomials and trigonometric polynomials were studied by Arnold in [2, 3], and for meromorphic functions on arbitrary Riemann surfaces by Vinberg [19], who introduced the nets. The dual graph of a net of a meromorphic function is known in classical function theory as a "line complex," or a Speiser graph [8, 20]. We use it in our proof of Lemmas 3 and 4.

Non-equivalent nets correspond to non-equivalent rational functions. We fix a net γ, and consider the map Φ from the space of labelings to the space of critical sets. Analyzing the boundary behavior of this map, and using a "continuity method," we show that Φ surjective. So for a given critical set, each γ gives a rational function of our class R, and it remains to count all possible classes of nets γ. It turns out that there are exactly u_d of them.

We thank Mario Bonk, who suggested the non-trivial normalization (5) of rational functions, and Boris Shapiro, for stimulating discussions of his conjectures.

We prove Theorem 1 only for $d \geq 3$, because it is trivial for $d = 2$, and because our proof would require a modification in this case.

We fix an integer $d \geq 3$. The map $z \mapsto 1/\bar{z}$ will be called the *symmetry*. A map will be called *symmetric* if it commutes with the symmetry. A set will be called *symmetric* if the symmetry leaves it invariant. All homeomorphisms and ramified coverings of the Riemann sphere $\overline{\mathbb{C}} = \mathbb{CP}^1$ are assumed to be orientation-preserving. For a Jordan region D we denote by ∂D its oriented boundary (so that the region is on the left). The unit circle $\mathbb{T}$ is always oriented anticlockwise, so $\mathbb{T} = \partial\mathbb{U}$, where $\mathbb{U}$ is the unit disc. The words "distance" and "diameter" refer to the spherical metric on the Riemann sphere. It is obtained from the standard embedding of $\overline{\mathbb{C}}$ as the unit sphere in $\mathbb{R}^3$. This metric induces the ordinary Euclidean metric on the unit circle $\mathbb{T}$.

2. Nets and their labelings

A *cellular decomposition* of a set $X \subset \overline{\mathbb{C}}$ is a finite partition of X into sets, called cells, each of them homeomorphic to an open unit disc $\mathbb{U}^k \subset \mathbb{R}^k$, $k = 0, 1, 2$; (by definition, $\mathbb{U}^0 = \{\text{one point}\}$), and has closure homeomorphic to the closed disc $\overline{\mathbb{U}}^k$. The cells are called vertices, edges and faces, according to their dimension. The degree of a vertex is the number of edges to whose boundaries this vertex belongs. A *net* $\gamma \subset \overline{\mathbb{C}}$ is the union of edges and vertices of some cellular decomposition of $\overline{\mathbb{C}}$, which satisfies conditions N1-N5 below.

N1. γ is symmetric,

N2. $\mathbb{T} \subset \gamma$,

N3. There are $2d-2$ vertices, all belonging to $\mathbb{T}$ and having degree 4.

N4. The point $1 \in \mathbb{T}$ is a vertex.

A cellular decomposition which satisfies N1-N4 is completely determined by its net γ, so we permit ourselves to speak of vertices, edges and faces *of a net*. Because of N3, each face G has even number of boundary vertices. For every γ satisfying N1-N4 we choose certain distinguished elements as follows. Let $v_0 = 1$, and v_1 the next vertex anticlockwise. There is a unique face G_0, whose boundary contains at least 4 vertices, v_0 and v_1 among them. Let v_{-1} be the vertex preceding v_0 on ∂G_0. So when tracing ∂G_0 according to its orientation, we consecutively encounter v_{-1}, v_0, v_1 in this order. We also introduce two edges on the boundary of G_0: $e_1 = [v_0, v_1]$ and $e_{-1} = [v_{-1}, v_0]$. One of these two edges, e' belongs to $\mathbb{T}$, another, e'' does not. Thus we have double notation for these two edges. For every γ satisfying N1-N4 there is a unique choice for the *distinguished elements* $G_0, v_{-1}, v_0, v_1, e_{-1}, e_1, e'$, and e''. Now the vertices of γ have natural ordering $v_1, \ldots, v_{2d-2}$, where $v_{2d-2} = v_0$, and $v_{-1} = v_N$, for some $N = N(\gamma) \in [3, 2d-3]$. Our last condition is the normalization

N5. $v_{-1} = e^{-2\pi i/3}, \quad v_0 = 1, \quad \text{and} \quad v_1 = e^{2\pi i/3}, \quad$ the cubic roots of 1.

(The particular choice of these three points on $\mathbb{T}$ is irrelevant). Two nets γ_1 and γ_2 are called *equivalent* if there exists a symmetric homeomorphism h of the sphere $\overline{\mathbb{C}}$, such that $h(\gamma_1) = \gamma_2$, and h leaves each cubic root of 1 fixed. Such h induces a bijective correspondence between the cells of the corresponding cellular decompositions, so we can speak of a vertex, an edge or a face of a class of nets. Each distinguished element described above is mapped by h onto a distinguished element with the same name. We denote by $[\gamma]$ the equivalence class of a net γ.

For a net γ we denote by V, E and Q the sets of its vertices, edges and faces, respectively. Euler's formula implies $|Q| = 2d$ and $|E| = 4d-4$. The

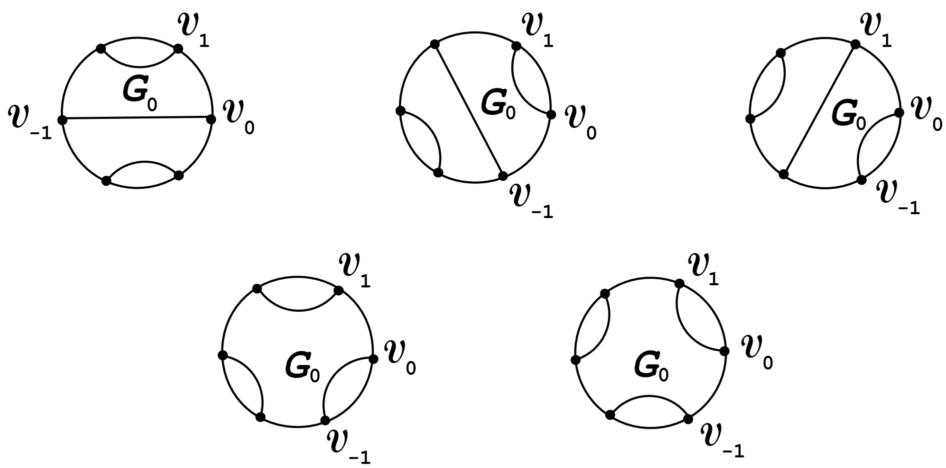


Figure 1: All nets for $d = 4$ (Only the parts in $\overline{\mathbb{U}}$ are shown).

subset $Q^+ \subset Q$ of faces which belong to $\mathbb{U}$, determines the net and the cellular decomposition completely.

Each class of nets has a *nice representative* $\gamma \subset \overline{\mathbb{C}}$, such that all edges of γ in the interior of the unit disc are spherical geodesics, whose closures we call *chords*.

Figure 1 shows all nets for $d = 4$ with distinguished faces and vertices. For aesthetic reasons we ignored N5 in this picture.

Lemma 1 *There exist exactly u_d classes of nets with $2d - 2$ vertices, where u_d is the Catalan number* (1)*.*

Proof. Because of the symmetry, γ is completely determined by its chords. So we have to solve the following problem: given $2n - 2$ distinct points on a circle, say $v_1, \ldots, v_{2n-2}$, enumerated in the natural cyclic order, find the number of ways u_n to draw $n - 1$ disjoint closed chords through all these points. To introduce a recurrence, we define $u_1 = 1$. We consider the chord passing through v_1. Let v_k be the other end of this chord. It is clear that k is even, so we set $k = 2m$. This chord separates the picture into two parts, one of them is a topological disc with $2m - 2$ boundary points, connected by disjoint chords, another is a topological disc with $2n - 2m - 2$ boundary points, connected by disjoint chords. From these considerations follows the recurrence relation

$$u_n = \sum_{m=1}^{n-1} u_m u_{n-m}.$$

Together with the initial condition $u_1 = 1$ this characterizes the Catalan numbers [12, p. 116]. □

For each net we define a function $\sigma : Q \to \{1, -1\}$, called *parity*. We put $\sigma(G_0) = 1$, for the distinguished face, and then $\sigma(G)\sigma(G') = -1$ if the faces G and G' have a common edge on their boundaries. Such parity function exists for every cellular decomposition whose vertices have even degree. With our normalization $\sigma(G_0) = 1$, the parity function is unique.

A *labeling* of a net is a non-negative symmetric function on the set of edges, $p : E \to \mathbb{R}$, satisfying the following two conditions:

$$\sum_{s \in \partial^* G} p(s) = 2\pi \quad \text{for every} \quad G \in Q^+, \tag{2}$$

where $\partial^* G$ is the set of non-oriented edges which belong to the boundary of G, and

$$p(e_1) = p(e_{-1}) = 2\pi/3. \tag{3}$$

A pair (γ, p) is called a *labeled net*. Two labeled nets (γ_1, p_1) and (γ_2, p_2) are *equivalent* if there exists a symmetric homeomorphism $h : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, fixing the three cubic roots of 1, and having the properties $h(\gamma_1) = \gamma_2$, and $p_2(h(e)) = p_1(e)$ for every edge e of γ_1.

A labeling p is called *degenerate* if $p(e) = 0$ for some edges $e \in E$, otherwise it is *non-degenerate*. The space of all labelings $\overline{L}_\gamma$ is a closed convex polytope in the affine subspace A of $\mathbb{R}^{4d-4}$, defined by (2), (3) and the symmetry condition. Its interior L_γ with respect to A, which is the set of non-degenerate labelings, is homeomorphic to a cell of dimension $2d - 5$,

3. Degenerate labelings

In this Section we study the structure of the set of degenerate labelings ∂L_γ. For $p \in \overline{L}_\gamma$ we define $Z(p)$ as the union of closed edges e of γ such that $p(e) = 0$, and $D(p)$ as the connected component of $\overline{\mathbb{C}} \backslash Z(p)$ containing G_0. Notice that $D(p)$ always contains at least 3 boundary edges of G_0, including

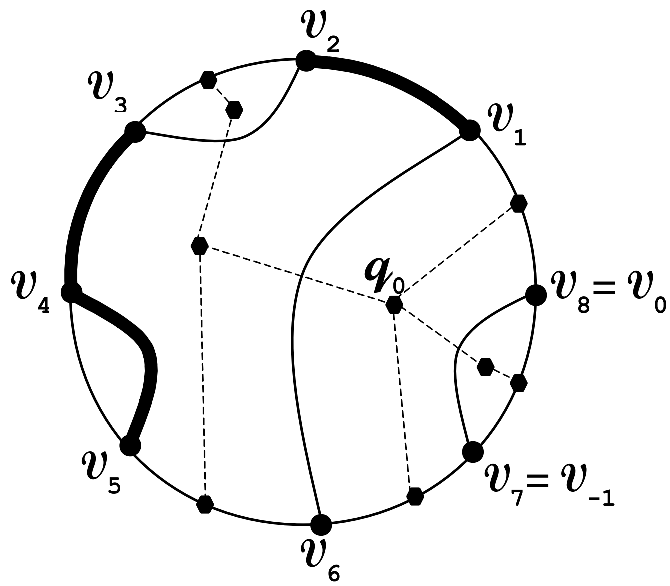


Figure 2: $S(p)$ in dotted lines, $E^0(p)$ in bold.

e_1 and e_2. This follows from (3) and (2) with $G = G_0$. Let $E^0(p)$ be the set of all edges of γ in $\partial D(p) \cap \overline{\mathbb{U}}$ and $E(p)$ the set of all edges in $D(p) \cap \mathbb{T}$.

Lemma 2 *For every $p \in \overline{L}_\gamma$, the set*

$$\overline{L}_\gamma(p) = \{q \in \overline{L}_\gamma : q(e) = 0 \quad \textit{for all} \quad e \in E^0(p)\}$$

is a non-empty face of $\overline{L}_\gamma$.

Proof. First, $\overline{L}_\gamma(p)$ is a face of $\overline{L}_\gamma$, because it is an intersection of the convex polytope $\overline{L}_\gamma$ with a set of hyperplanes $\{q : q(e) = 0\}$, with $\overline{L}_\gamma \subset \{q : q(e) \geq 0\}$, for all $e \in E^0(p)$. Second, $\overline{L}_\gamma(p)$ is nonempty because it contains p. □

Lemma 3 *For $p \in \overline{L}_\gamma$, the set $E^0(p)$ and the face $\overline{L}_\gamma(p)$ are uniquely determined by the set $E(p)$.*

Proof. Let S be the dual graph to $\gamma \cap \overline{\mathbb{U}}$, which means that each vertex $q = q_G$ of S corresponds to a face $G = G_q \in Q^+$, and two vertices of S are connected by an edge $\tau = \tau_e$ in S if the two corresponding faces of γ have a common edge $e = e_\tau$ in γ.

Let $\hat{S}$ be the graph obtained by the following extension of S: for every edge $e \subset \mathbb{T}$ of γ, a vertex q_e and an edge τ_e connecting q_e with q_G are added to S, where G is the face in Q^+ with $e \in \partial^* G$.

It is easy to see that $\hat{S}$ is a tree. We designate $q_0 = q_{G_0}$ to be the root of this tree. Let $S(p)$ be the subtree of $\hat{S}$ spanned by q_0 and $\{q_e : e \in E(p)\}$, and $S'(p)$ the subtree of $\hat{S}$ spanned by $\{q_x : x \subset D(p)\}$ (where x may stand for a face or an edge). By definition of $E(p)$, we have $S(p) \subset S'(p)$. We claim that $S(p) = S'(p)$, which means that $D(p) \cap \overline{\mathbb{U}}$ consists of all faces G of $\gamma \cap \overline{\mathbb{U}}$ such that $q_G \in S(p)$. Thus $E(p)$ uniquely determines $D(p)$, and $D(p)$ uniquely determines $E^0(p)$ and $\overline{L}_\gamma(p)$. Figure 2 shows the part in $\overline{\mathbb{U}}$ of a net γ with $d = 5$, the set $E^0(p) = \{[v_1, v_2], [v_3, v_4], [v_4, v_5]\}$ (bold lines) and the tree $\hat{S}$ (dotted lines).

To prove our claim, suppose that $S'(p) \not\subset S(p)$. Since both $S(p)$ and $S'(p)$ belong to the tree $\hat{S}$, there exists a leaf q of $S'(p)$ which does not belong to $S(p)$. If $q = q_e$, where $e \subset \mathbb{T}$, is an edge of γ, then $e \subset D(p)$, hence q_e is a vertex of $S(p)$, in contradiction to our choice of q. Suppose now that $q = q_G$, where $G \subset D(p)$, is a face in Q^+. Let S_q be be the path in S connecting q and q_0. Conditions (2) imply that $0 < p(e) < 2\pi$ for every $e = e_\tau$, $\tau \in S_q$. This implies that there is an edge e of $\partial^* G$ such that $\tau_e \notin S_q$ and $0 < p(e) < 2\pi$. Since $q \notin S(p)$, we have $\tau_e \notin S(p)$. If $e \subset \mathbb{T}$, we have a contradiction with the definition of $E(p)$. Otherwise, the other face of $\gamma \cap \overline{\mathbb{U}}$, having the edge e on its boundary, belongs to $D(p)$, and G is not a leaf of $S'(p)$, again a contradiction. □

Lemma 4 *The set $W = E(p)$ has the following two properties:*

(a) *$e' \in W$;*

(b) *there is at least one edge in $W \setminus \{e'\}$ at each side of e''.*

Proof. Let $S(p)$ be the tree defined in the proof of Lemma 3. The leaves of $S(p)$ are exactly the edges in $E(p)$. One of these leaves is always $\tau_{e'}$. Since G_0 has at least three edges, and since $p(e') = p(e'') = 2\pi/3$ in view of (3), there exists at least one edge e^* of G_0 different from e' and e'' such that $0 < p(e^*) < 2\pi$. Hence there are at least two edges of $S(p)$ with an extremity at $q_0 = q_{G_0}$, corresponding to e'' and e^*. The other extremities of these two edges are at the opposite sides of e''. Hence $S(p)$ has a least two leaves on $\mathbb{T}$ in addition to $\tau_{e'}$, and these two leaves are at the opposite sides of e''. □

Lemma 5 *For each subset W of edges of $\gamma \cap \mathbb{T}$ satisfying* (a) *and* (b) *of Lemma 4, there is $p \in \overline{L}_\gamma$ such that $W = E(p)$.*

Proof. Given a subset W of edges of $\gamma \cap \mathbb{T}$ satisfying (a) and (b), let us define a subtree S_W of the tree $\hat{S}$ defined in the proof of Lemma 3 as the union of all paths connecting vertices q_e, for $e \in W$ with q_0. The labeling p is defined inductively along the tree S, starting from the vertex q_0. As W contains, in addition to e', at least one edge at each side of e'', we have $\tau_{e''} \subset S_W$, and there is at least one edge e of G_0, other than e' and e'' such that $\tau_e \in S_W$. Let $m \geq 1$ be the number of all such edges. We define $p(e) = 2\pi/(3m)$ for each of them, and $p(e) = 0$ for all other edges of G_0, except $p(e') = p(e'') = 2\pi/3$. This guarantees that (2) is satisfied for G_0. Notice that $0 < p(e) < 2\pi$ for an edge e of $\partial^* G_0$ if and only if $\tau_e \in S_W$.

Suppose now that the values of $p(e)$ are defined for all edges of faces $G_q \in Q^+$, with q in a subtree S' of S containing q_0, so that $0 < p(e) < 2\pi$ if and only if τ_e belongs to S_W, and (2) is satisfied. If $S' = S$, the labeling p is complete. Otherwise, there exists a vertex q^* in $S \setminus S'$ which is an extremity of an edge τ^* of S with another extremity of τ^* being in S'. Let $G^* = G_{q^*}$ and $e^* = e_{\tau^*}$. Since an extremity of τ^* belongs to S', the label $p(e^*)$ is already defined.

If $p(e^*) = 2\pi$ or $p(e^*) = 0$, then e^* does not belong to S_W, hence all other edges of G^* do not belong to S_W. In the first case, we define $p(e) = 0$ for all edges $e \neq e^*$ of G^*. In the second case, we choose an edge $e^{**} \neq e^*$ of G^* and define $p(e^{**}) = 2\pi$ and $p(e) = 0$ for all other edges of G^*. Then (2) is satisfied for $G = G^*$.

If $0 < p(e^*) < 2\pi$, then τ^* belongs to S_W. Since $e^* \notin \mathbb{T}$, there is at least one other edge e of G^* such that τ_e belongs to S_W. Let $n \geq 1$ be the number of all such edges. We define $p(e) = (2\pi - p(e^*))/n$ for all these edges, and $p(e) = 0$ for all other edges $e \neq e^*$ of G^*. Again we have (2) for $G = G^*$.

Now the values of $p(e)$ are defined for all edges of faces $G_q \in Q^+$, for the vertices q of a connected subtree S'' of S obtained by adding τ^* and q^* to S', which concludes our inductive step. The labeling p constructed in this way satisfies (2) and that $W = E(p)$. $\square$

A *non-degenerate critical sequence* corresponding to γ, is an injective map $c : V \to \mathbb{T}$, which leaves v_{-1}, v_0 and v_1 fixed, and preserves the cyclic order. The set of all non-degenerate critical sequences is identified with an open convex polytope $\Sigma_\gamma \subset \mathbb{R}^{2d-5}$. A *critical sequence* is a limit of non-degenerate critical sequences. The set of all critical sequences is identified with the closure $\overline{\Sigma}_\gamma$.

We denote by R the class of all rational functions of degree at most d, which preserve the unit circle, and whose critical points all belong to the unit circle, and satisfy the normalization condition

$$f(z)=z, \quad f'(z)=0, \quad \text{for} \quad z\in\{1,e^{2\pi i/3},e^{-2\pi i/3}\}. \tag{4}$$

If two functions f_1 and f_2 of the class R are equivalent, that is $f_1=\ell\circ f_2$, with a fractional-linear transformation ℓ, then ℓ preserves $\mathbb{T}$. Equivalent functions of our class R have *the same* preimage of the unit circle.

For each net γ we consider a subclass $R_\gamma\subset R$ defined by the following additional normalization condition:

$$f^{-1}(\mathbb{T}) \quad \text{is equivalent to } \gamma. \tag{5}$$

It follows from (5) that all critical points of $f\in R_\gamma$ are simple, and f maps the distinguished face G_0 onto the unit disc.

4. Construction of a map

$$F_\gamma : \overline{L}_\gamma \to R\times\overline{\Sigma}_\gamma. \tag{6}$$

We consider a labeling $p\in\overline{L}_\gamma$. We use the notation similar to that introduced in the first paragraph of Section 3, but because γ and p are fixed, we drop the reference to p and γ to simplify our formulas. Thus Z is the union of edges with zero labels, and D the component of $\overline{\mathbb{C}}\backslash Z$, containing G_0. We put $K=\overline{\mathbb{C}}\backslash D$ and introduce an equivalence relation in $\overline{\mathbb{C}}$: $x\sim y$ if $x=y$, or x and y belong to the same component of K. Let $Y=\overline{\mathbb{C}}/\sim$ be the factor space, and $w:\overline{\mathbb{C}}\to Y$ the projection map. It is clear that X is a topological sphere, so we can identify it with the Riemann sphere. The symmetry of $\overline{\mathbb{C}}$ is an involution which leaves every point of $\mathbb{T}$ fixed, and each component of K is symmetric. So Y also has an involution, such that w splits the involutions. This means that the identification of Y with $\overline{\mathbb{C}}$ can be made in such a way, that

$$w:\overline{\mathbb{C}}\to Y\cong\overline{\mathbb{C}}, \quad w(x)=w(y) \quad \text{if and only if} \quad x\sim y, \tag{7}$$

is symmetric, and in particular $w(\mathbb{T})=\mathbb{T}$. Furthermore, we can arrange that w leaves the cubic roots of 1 fixed. The cellular decomposition of $\overline{\mathbb{C}}$ defined by γ generates via w a new cellular decomposition, which we call $X = X(p)$: the cells of X are $w(C)$, where C are cells of the original decomposition.

We construct a continuous map $g^* : \overline{D} \to \overline{\mathbb{C}}$. As a first step of our construction of g^*, we define a continuous map $\tilde{g} : \gamma \cap \overline{D} \to \mathbb{T}$. To do this, we orient the edges of γ in the following way. Each edge $e \in E$ belongs to the boundaries of exactly two faces; let G be that one with $\sigma(G) = 1$. Then $e \subset \partial G$ by definition inherits positive orientation of ∂G.

We are going to define $\tilde{g}$, so that the following condition be satisfied:

$$\begin{gathered}\textit{$\tilde{g}$ maps every edge $e \subset \overline{D}$ onto an arc of $\mathbb{T}$ of length $p(e)$,}\\ \textit{linearly with respect to the arclength, respecting orientation,}\end{gathered} \tag{8}$$

in particular the edges in ∂D are mapped into points, but closures of all edges in D are mapped homeomorphically onto their images.

First we define $\tilde{g}$ on ∂G_0, so that (8) is satisfied, and $\tilde{g}(v_0) = 1$. Condition (2) with $G = G_0$ ensures that there is a unique way to define such continuous $\tilde{g}$ on ∂G_0. Furthermore, (3) implies that $\tilde{g}$ fixes all three cubic roots of 1.

Now we order all faces of γ in D into a sequence $(G_0, G_1, \ldots, G_m)$ so that for every $k = 1, \ldots, m$ the face G_k has exactly one common boundary edge with

$$\bigcup_{j=0}^{k-1} \partial^* G_j. \tag{9}$$

Such ordering can be made, for example, using the dual graph S, introduced in the proof of Lemma 3. Suppose that $\tilde{g}$ has been already defined on the edges in (9). Condition (2) with $G = G_k$ implies that there exists a continuous map $\tilde{g} : \partial G_k \to \mathbb{T}$, satisfying (8). This map is defined up to a rotation of the image $\mathbb{T}$. We choose this rotation to ensure that $\tilde{g}$ is continuous on

$$\bigcup_{j=0}^{k} \partial G_j.$$

This is possible, because $\partial^* G_k$ has exactly one common edge with the collection (9). This construction defines a symmetric continuous map $\tilde{g} : \gamma \cap \overline{D} \to \mathbb{T}$, which sends every component of ∂D to a point.

As a next step, for each face $G \subset D$, we extend $\tilde{g}$ to a continuous map $g^* : \overline{G} \to \overline{\mathbb{U}}$, if $\sigma(G) = 1$, or $g^* : \overline{G} \to \overline{\mathbb{C}} \backslash \mathbb{U}$, if $\sigma(G) = -1$, so that the restriction on G is a homeomorphism onto the image.

It is clear, that this extension of g into the interior of components $G \in Q$, $G \subset D$, can be made symmetrically, that is

$$g^*(1/\overline{z}) = 1/\overline{g^*(z)}, \quad z \in \overline{\mathbb{C}}. \tag{10}$$

Finally we extend g^* to a continuous map $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$ so that it is constant on every component of the set K. Then $g^*(x) = g^*(y)$ whenever $x \sim y$, the equivalence relation $\sim$ in (7). It follows that g^* factors as $g^* = g \circ w$, where w is the continuous map in (7).

If C is a cell of the cellular decomposition defined by γ, then w and g^* map C in the same way: either homeomorphically or to a point. It follows that g maps every closed cell of the form $w(\overline{C})$ homeomorphically onto the image. Furthermore, the cells $w(C)$ make a cellular decomposition X of Y, so g is a ramified covering. Thus there exists a unique conformal structure on Y, which makes g holomorphic. By the Uniformization theorem [1] there exists a unique homeomorphism $\phi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, normalized by

$$\phi(e^{-2\pi i/3}) = -2\pi i/3, \quad \phi(1) = 1, \quad \phi(e^{2\pi i/3}) = e^{2\pi i/3}, \tag{11}$$

and such that $f = g \circ \phi^{-1}$ is a holomorphic map $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$, that is a rational function. This function is the first component of $F_\gamma(p)$. The second component is $c : v \mapsto \phi \circ w(v)$, $v \in V$, which is a critical sequence in $\overline{\Sigma}_\gamma$. Indeed, by the symmetry property (10) and the symmetry of the normalization (11), ϕ is symmetric. Applying (10) again, we conclude that our rational function f is symmetric, and that all values of the function c belong to $\mathbb{T}$. We write $c = (c_1, \ldots, c_{2d-2})$, where $c_k = \phi \circ w(v_k)$. Then it follows from our constructions that

$$c_k \neq c_{k+1} \quad \text{if and only if the edge} \quad [v_k, v_{k+1}] \quad \text{belongs to} \quad D \cap \mathbb{T}. \tag{12}$$

Now we show that our map (6) is well defined, that is f and c do not depend on the choice of a labeled net within its equivalence class, and also independent of extensions of g into the interiors of the components G. This independence follows from

Lemma 6 *Let X_i, $i = 0, 1$, be cell complexes, h' a bijection between their cells such that $h'(\partial^* C) = \partial^* h'(C)$, Y a topological space and $f_i : X_i \to Y$ two continuous maps, whose restrictions to every closed cell are homeomorphisms onto the image, and $f_1(C) = f_0(h'(C))$ for every cell C in X_1. Then there exists a homeomorphism h such that $f_1 = f_0 \circ h$.*

Proof. We define h on every cell C in X_1 as $f^{-1}_{0,h'(C)} \circ f_1|_C$, where $f^{-1}_{0,h'(C)}$ is the inverse of the restriction $f_0|_{h(C)} : h(C) \to f_0(h(C))$. □

Applying Lemma 6 to two rational functions f_0 and f_1, constructed from equivalent labeled nets, we conclude that $f_0 = f_1 \circ h$, where h is a homeomorphism of the Riemann sphere. This homeomorphism is evidently conformal and fixes three points. So $h = \mathrm{id}$ and $f_1 = f_0$.

Lemma 7 *The map* $([\gamma], p) \mapsto c$ *is well defined, that is* $c = F_\gamma(p)$ *depends only on the class of labeled nets* $([\gamma], p)$.

Proof. Consider the cellular decomposition X, introduced after equation (7). If v is a vertex of X of degree at least 4, then $z = \phi(v)$ is a critical point of f, so $c(v)$ is well defined. Suppose now that $v^1, \ldots, v^m$ is a maximal chain of vertices of X of degree 2, which means that there are edges in X between these vertices, but no other edges connecting v^1 or v^m to vertices of degree 2. There is a unique way to extend this chain by adding v^0 and v^{m+1}, vertices of degree at least 4, so that v^0 is connected to v_1 and v^m to v^{m+1} by edges of X. Then $z^j = \phi(v^j),\ j = 0, m+1$, are critical points of f, and $a_j = f(z^j),\ j = 0, m+1$, corresponding critical values. The restriction of f onto the arc $[z^0, z^{m+1}] \subset \mathbb{T}$ maps this arc homeomorphically onto the arc $[a_0, a_{m+1}] \subset \mathbb{T}$. Then the position of the points $z^j = \phi(v_j),\ j = 1, \ldots, m$, is determined from the fact that the length of each arc $[f(z^k), f(z^{k+1})] \subset \mathbb{T}$ is equal to $p(w^{-1}([v^k, v^{k+1}]))$, the label of an edge of γ. □

5. Continuity

For a fixed γ, the second coordinate of our map F_γ in (6) is a map between two closed polytopes

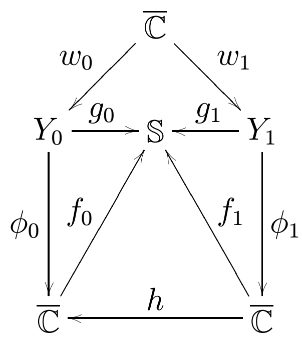
$$\Phi : \overline{L} \to \overline{\Sigma}, \tag{13}$$

where $L = L_\gamma$ and $\Sigma = \Sigma_\gamma$. Our goal is to prove that Φ in (13) is surjective. In this Section we prove that Φ is continuous.

Suppose $p_1 \in \overline{L}$; we are going to prove that Φ is continuous at p_1. Let p_0 be a point close to p_1. Using the notation, similar to that introduced in the first paragraph of Section 3, we consider the sets Z_i and the regions $D_i,\ i = 0, 1$. In addition, let $B_1, \ldots, B_m$ be the complete list of components of ∂D_1. Then $B_k \subset Z_1$ for $k = 1, \ldots, m$. If p_0 is close enough to p_1, we may assume

$$Z_0 \subset Z_1, \quad \text{and thus} \quad D_1 \subset D_0. \tag{14}$$

We have the maps $w_i : \overline{\mathbb{C}} \to Y_i \cong \overline{\mathbb{C}},\ i = 0, 1$ as in (7), and $f_i = g_i \circ \phi_i^{-1},\ g_i^* = g_i \circ w_i,\ i = 0, 1$, defined in Section 4. All maps involved in our argument are shown on the diagram below, where we use double notation $\mathbb{S} = \overline{\mathbb{C}}$ for clarity. For every vertex $v \in V$ the "critical values" $a(v) = g_1 \circ w_1(v)$ are defined, $a_k = a(v_k) \in \mathbb{S},\ k = 1, \ldots, 2d - 2$.



We choose arbitrary $\delta > 0$. Then there exists $\epsilon > 0$, such that disjoint open discs U_k of radii ϵ around a_k have the property, that every component K of the preimage of their union under f_1 has diameter less than δ. We set $\epsilon_1 = \epsilon/(8d)$ and suppose that

$$|p_1(e) - p_0(e)| < \epsilon_1 \quad \text{for every} \quad e \in E \tag{15}$$

in particular, in view of (14),

$$|p_0(e)| < \epsilon_1 \quad \text{for} \quad e \subset \bigcup_{k=1}^{m} B_k. \tag{16}$$

The set

$$H = \mathbb{S} \backslash \bigcup_{k=1}^{q} U_k \tag{17}$$

has a cell decomposition with two 2-dimensional cells C and C^*, where $\overline{C} = \overline{\mathbb{U}} \backslash \cup_{k=1}^{q} U_k$, and C^* the symmetric cell. We choose 1-dimensional cells of this decomposition to be arcs of the unit circle and arcs of the circles ∂U_k, and for 0-dimensional cells we take the points of intersections of little circles with the unit circle.

Let H_1 be the preimage of the set H in (17) under f_1. Then H_1 has a cell decomposition, which is the preimage of our cell decomposition of (17), and $f_1|_{H_1}$ maps every cell of this decomposition homeomorphically. In fact $f_1|_{H_1}$ is an unramified covering map, because f_1 has no no critical in H_1.

It follows from (16) that

$$\text{diam}\, f_0 \circ \phi_0 \circ w_0(B_k) < \epsilon/2, \quad \text{for each component } B_k \text{ of } \partial D_1, \tag{18}$$

because B_k is made of at most $4d-4$ edges, whose labels are at most ϵ_1 each. Furthermore, we have

$$\text{dist}\,(f_0 \circ \phi_0 \circ w_0(B_k), a_k) < \epsilon, \quad \text{for each component } B_k \text{ of } \partial D_1, \tag{19}$$

which follows from (15) and (18).

Let H_0 be the component of $f_0^{-1}(H)$, where H is defined in (17), which contains $\phi_0 \circ w_0(G_0)$. In view of (19), the restriction $f_0|_{H_0} : H_0 \to H$ is a ramified covering. (Actually it is not ramified, but we don't use this fact). Thus the cell decomposition of H defined above pulls back to H_0, and f_0 maps each closed cell of this pullback onto a cell in H homeomorphically. Notice that each open cell of H_0 is contained in a unique cell of the form $\phi_0 \circ w_0(C)$ for some cell $C \subset D$ of γ. A similar statement holds for cells H_1. This defines a bijection between cells of H_1 and those of H_0 which commutes with the boundary operator ∂. So Lemma 6 can be applied to $f_i|_{H_i}$. We conclude that

$$f_1 = f_0 \circ h \quad \text{on} \quad H_1, \tag{20}$$

where $h : H_1 \to H_0$ is a homeomorphism. Evidently h is holomorphic, and its boundary values on ∂H_1 belong to ∂H_0. Moreover, the components of ∂H_1, which separate H_1 from the cubic roots of 1, are mapped to components of ∂H_0, which separate the same cubic roots of 1 from H_0. Now we use the following

Lemma 8 *Suppose that a finite set $X = \{x_1, x_2, \ldots, x_n\} \subset \overline{\mathbb{C}}$, is given, such that $n \geq 3$, and x_1, x_2 and x_3 are the cubic roots of 1. Then for every $\eta > 0$ there exists $\delta \in (0, \eta)$ with the following property. Let $J_1, \ldots, J_n$ be disjoint open Jordan regions of diameter less than δ, $x_k \in J_k$, $k = 1, \ldots, n$, and h be an injective holomorphic function*

$$h : \overline{\mathbb{C}} \backslash \bigcup_{k=1}^{n} J_k \to \overline{\mathbb{C}},$$

such that for $j = 1, 2, 3$ the curves $h(\partial J_j)$ separate x_j from the two other cubic roots of 1*. Then*

$$\mathrm{dist}\,(h(z), z) < \eta, \quad \textit{whenever} \quad \mathrm{dist}\,(z, X) \geq \eta.$$

Proof. (Compare [4, Theorem 13]). Our proof is by contradiction. Suppose that there is a sequence $(\delta_j) \to 0$ and a sequence (h_j), which satisfies all conditions, but

$$\mathrm{dist}\,(h_j(z_j), z_j) \geq \eta \tag{21}$$

for some $\eta > 0$ and some points z_j with $\mathrm{dist}\,(z_j, X) \geq \eta$. It is easy to see that the closed domains $R_j = \overline{\mathbb{C}} \setminus \cup_{k=1}^{n} J_{k,j}$ tend to $\overline{\mathbb{C}} \setminus \{x_1, \ldots, x_n\}$ and that all functions h_j omit three cubic roots of 1 in their domains. So by Montel's criterion [13], [14, p. 239], (h_j) is a normal family and we can select a convergent subsequence. The limit h of this subsequence is a holomorphic injective function h in $\overline{\mathbb{C}} \setminus \{x_1, \ldots, x_n\}$, which omits the three cubic roots of 1. By the Picard's great theorem all points x_k are removable singularities, so h extends to a fractional-linear map. But this fractional-linear map also fixes three points, the cubic roots of 1, so it is the identity. Ths contradicts (21). □

It follows that the critical sequences $(\phi_1 \circ w_1(v))$, $v \in V$ and $(\phi_0 \circ w_0(v))$, $v \in V$ are η-close. So our map (13) is continuous.

6. Surjectivity of Φ and proof of Theorem 2

Lemma 9 *Let L and Σ be two convex n-dimensional polytopes. Let $\Phi : \overline{L} \to \overline{\Sigma}$ be a continuous mapping such that preimage of each face of $\overline{\Sigma}$ is a nonempty face of $\overline{L}$. Then Φ is surjective.*

Proof. Let $C_k(\overline{L})$ be the space of k-chains of $\overline{L}$, i.e., linear combinations of oriented k-faces of $\overline{L}$ with integer coefficients. Let C_* be the corresponding chain complex, with the natural differential $\partial : C_k(\overline{L}) \to C_{k-1}(\overline{L})$. As $\overline{L}$ is a convex polytope, $H_k(C_*(\overline{L})) = 0$ for all positive k. Let $C_*(\overline{\Sigma})$ be the corresponding chain complex for $\overline{\Sigma}$.

For every k-face M of $\overline{\Sigma}$, we are going to construct a chain $W = W_M \in C_k(\overline{L})$ so that $\Phi(\partial W) \subset \partial M$ and $\Phi_*[W] = [M]$. Here Φ_* is the mapping $H_k(\Phi^{-1}M, \Phi^{-1}\partial M) \to H_k(M, \partial M)$ induced by Φ, $[W]$ is the class of W in $H_k(\Phi^{-1}M, \Phi^{-1}\partial M)$, and $[M]$ is the class of M in $H_k(M, \partial M) \cong \mathbb{Z}$. For $k =$

n, every chain in C_n is a multiple of $\overline{L}$, hence $\Phi(\partial L) \subset \partial\Sigma$ and $\Phi_*[\overline{L}] = \pm[\overline{\Sigma}]$. This implies that the induced mapping of $(n-1)$-spheres $\overline{L}/\partial L \to \overline{\Sigma}/\partial\Sigma$ has degree ± 1, in particular it is surjective. Hence, Φ maps L onto Σ.

We proceed inductively on $k = \dim M$. For $k = 0$, preimage of the vertex M is a nonempty face K of $\overline{L}$. Let W_M be a vertex of K. Suppose that the chains W_M are defined for all M with $\dim M < k$, so that $\partial W_M = W_{\partial M}$. Here a chain W_C for a chain $C = \sum m_\nu M_\nu$ is defined as $\sum m_\nu W_{M_\nu}$.

Let $M \in C_k(\overline{\Sigma})$. Due to the induction hypothesis, $W_{\partial M}$ is a cycle in $C_k(\overline{L})$, and $\Phi_*[W_{\partial M}] = [\partial M]$. Here Φ_* is the mapping $H_{k-1}(\Phi^{-1}\partial M) \to H_{k-1}(\partial M)$ induced by Φ. As $\Phi^{-1}M$ is a convex polytope, there exists a chain $W_M \in C_k(\Phi^{-1}M)$ such that $\partial W_M = W_{\partial M}$. From the commutative diagram

$$\begin{array}{ccc} H_k(\Phi^{-1}M) & \xrightarrow{\partial} & H_{k-1}(\Phi^{-1}\partial M) \\ {\scriptstyle \Phi_*}\downarrow & & {\scriptstyle \Phi_*}\downarrow \\ H_k(M) & \xrightarrow{\partial} & H_{k-1}(\partial M) \end{array}$$

we have $\partial\Phi_*[W_M] = \Phi_*\partial[W_M] = [\partial M]$. As $H_k(M) \cong \mathbb{Z}$ is generated by $[M]$ and $\partial[M] = [\partial M]$, this implies $\Phi_*[W_M] = [M]$. □

Proof of Theorem 2. First we verify that our map Φ in (13) satisfies conditions of Lemma 9. In Section 5 we established that Φ is continuous. Let us consider now preimages of the faces. We write a critical sequence $c = (c_k) = (c(v_k))$ as $c_k = \exp(i\theta_k)$, where

$$2\pi i/3 = \theta_1 < \theta_2 \leq \ldots \leq \theta_{2n-2} = 2\pi i, \quad \text{and} \quad \theta_N = 4\pi/3. \tag{22}$$

Here $N = N(\gamma)$ is the integer, introduced in Section 2, with the property that $v_{-1} = v_N$, $3 \leq N \leq 2d-3$. A face $\Sigma_\gamma(c)$ in Σ_γ, containing c is described by telling, which inequalities in (22) are strict. There is at least one strict inequality between θ_1 and θ_N, and at least one between θ_N and θ_{2d-2}. According to (12) we have $\theta_k < \theta_{k+1}$ if and only if the edge $[v_k, v_{k+1}]$ belongs to $D(p)$. Thus the sequence (22) defines a subset W of edges of γ, which satisfies conditions of Lemma 5. By Lemma 3 this W defines some labeling p with $W = E(p)$. According to Lemmas 3 and 2 E_p determines a unique non-empty face $\overline{L}_\gamma(p)$. This face is the preimage of the face $\Sigma_\gamma(c)$.

Thus all conditions of Lemma 9 are satisfied, so Φ is surjective. This implies that for every fixed non-degenerate critical sequence c and every γ there exists a rational function $f_\gamma \in R_\gamma$, which has c as its sequence of critical points. Now we notice that for functions in R_γ with $2d-2$ simple critical points, the net γ is equivalent to $f_\gamma^{-1}(\mathbb{T})$. Thus if two such functions f_{γ_1} and f_{γ_2} are equivalent, then γ_1 and γ_2 are equivalent. By Lemma 1 there is u_d of classes of nets, so there are u_d classes of functions with a given critical set c. $\square$

7. Proof of Theorem 1

In this Section we derive Theorem 1 from Theorems A and 2. The vector space of non-zero polynomials of degree at most d with complex coefficients is naturally identified with $\mathbb{C}^{d+1} =: \mathrm{Poly}_d$. Every pair (r, q) of non-proportional polynomials spans a 2-dimensional subspace in Poly_d.

To parametrize the equivalence classes of rational functions of degree d, we consider the Grassmanian $G(2, d+1)$, which is the set of all 2-dimensional subspaces in Poly_d, and the locus $D_1 \subset G(2, d+1)$ of those pairs of polynomials (r, q), for which $\deg r/q < d$. Then D_1 is an algebraic subvariety of $G(2, d+1)$ of codimension 1. Two pairs $(r_i, q_i),\ i = 1, 2$, represent the same point in $G(2, d+1)\backslash D_1$ if and only if the rational functions r_1/q_1 and r_2/q_2 are equivalent. Thus classes of rational functions of degree d are parametrized by $G(2, d+1)\backslash D_1$.

The Wronskian determinant of two non-proportional polynomials

$$W(r, q) = \begin{vmatrix} r & q \\ r' & q' \end{vmatrix}$$

is a non-zero polynomial of degree at most $2d-2$, whose zeros are finite critical points of $f = r/q$, counting multiplicities, and common zeros of r and q. The common zeros of r and q are multiple zeros of $W(r, q)$. If two pairs of polynomials define the same point in $G(2, d+1)$, then the Wronskians of these pairs differ by a constant multiple. The set of all non-zero polynomials of degree at most $2d-2$, modulo proportionality, is parametrized by $\mathbb{CP}^{2d-2}$. Thus we have a regular map $\widetilde{W} : G(2, d+1) \to \mathbb{CP}^{2d-2}$, defined by taking the proportionality class of the Wronskian determinant.

We show that $\widetilde{W}$ is a finite map [10, p. 177]. This fact is known, [5, 9] but we include a short proof. We normalize our Wronskians, so that the coefficient of the monomial of the smallest degree equals 1. Notice that each monomial z^n, where $0 \leq n \leq 2d-2$, has only finitely many preimages under $\widetilde{W}$, namely the 2-subspaces, generated by pairs (z^k, z^m), where $k+m = n+1$ and $k \neq m$. If (r, q) represents a point in $G(2, d+1)$, we consider the one-parametric family of points represented by (r_λ, q_λ), $\lambda \in \mathbb{C}^*$, where $r_\lambda(z) = r(\lambda z)$ and $q_\lambda(z) = q(\lambda z)$. Putting $w_\lambda = \widetilde{W}(r_\lambda, q_\lambda)$, we obtain $W(r_\lambda, q_\lambda)(z) = \lambda W(r, q)(\lambda z)$, and after normalization $w_\lambda(z) = \lambda^{-n-1} W(r, q)(\lambda z)$, where $n \in [0, 2d-2]$ is the smallest degree of monomials in $W(r, q)$. So $\dim \widetilde{W}^{-1}(w_\lambda) = \dim \widetilde{W}^{-1}(w_1)$ for $\lambda \in \mathbb{C}^*$, and

$$w_0 := \lim_{\lambda \to 0} w_\lambda \quad \text{is} \quad w_0(z) = z^n.$$

As the dimension of preimage is an upper semi-continuous function of the point [10, p. 138], for regular mappings into compact spaces, that is

$$\limsup_{\lambda \to 0} \dim \widetilde{W}^{-1}(w_\lambda) \leq \dim \widetilde{W}^{-1}(w_0) = 0,$$

we conclude that $\dim \widetilde{W}^{-1}(w_1) = 0$ for every $w_1 \in \mathbb{CP}^{2d-1}$, so the preimages are finite, and the map $\widetilde{W}$ is finite.

Let $D_2 \subset \mathbb{CP}^{d-2}$ be the locus of polynomials with multiple roots, or having smaller degree than $2d-2$. Notice that $\widetilde{W}(D_1) \subset D_2$. According to Theorem A, for every point w in $\mathbb{CP}^{2d-2} \backslash D_2$ we have $|\widetilde{W}^{-1}(w)| \leq u_d$. On the other hand, our Theorem 1 implies that for every point w in $\mathbb{RP}^{2d-2} \backslash D_2 \subset \mathbb{CP}^{2d-2} \backslash D_2$ the cardinality of $\widetilde{W}^{-1}(w) \cap G_{\mathbb{R}}(2, d+1) \subset G(2, d+1)$ is at least u_d. Here $G_{\mathbb{R}}(2, d+1)$ stands for the 'real part' of the Grassmanian, that is the collection of those 2-dimensional subspaces which can be generated by pairs of real polynomials. This means that

$$\widetilde{W}^{-1}(\mathbb{RP}^{2d-2} \backslash D_2) \subset G_{\mathbb{R}}(2, d+1).$$

On the other hand, for finite maps we have $\widetilde{W}^{-1}(w) = \lim_{w' \to w} \widetilde{W}^{-1}(w')$, so $\widetilde{W}^{-1}(\mathbb{RP}^{2d-2}) \subset G_{\mathbb{R}}(2, d+1)$. □

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