# F. Gehring's problem on linked curves

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Let $A$ and $B$ be two closed curves in $R^3$ which are linked[1]. Suppose that the distance between $A$ and $B$ is 1. Prove that length of $A$ (or $B$) is at least $2\pi$.

*First proof.* (Sketch of the proof by A. E. and O. Vinkovskii). Suppose that length $B < 2\pi$.

Let $F$ be the surface of minimal area whose boundary is $B$. (A continuous map from the closed unit disc to $R^3$ such that the image of the unit circle is $B$, and which is minimizing the area). Then $A$ must intersect $F$; let $O$ be a point of intersection. Consider the sphere $S$ of unit radius centered at $O$. The curve $B$ must be outside this sphere. Let $B'$ be the central projection from $O$ of $B$ on $S$. Then length $B' \leq$ length $B < 2\pi$. So $B'$ belongs to an open hemisphere (see, for example [1, 6.1.1]). This implies that $B'$ and $B$ belong to an open half-space which has $O$ on its boundary. This contradicts the minimality of the area of $F$.

I told this proof on the geometric seminar of I. N. Pesin in Lvov University in 1976 or 1977. A day later, one of the participants, a second-year undergraduate student I. Syutrik told me the following proof:

*Second proof* (I. Syutrik). Fix a point $M$ on $A$. Then one can find another point $M'$ on $A$ such that the interval $[M, M']$ intersects $B$. Indeed, otherwise we can deform $A$ to $M$ moving straight along these intervals $[M, M']$ and deformation will not cross $B$. Let $O$ be a point on $[M, M']$ that belongs to $B$. Let $A'$ be the central projection of $A$ from $O$ onto the unit sphere around $O$. Then $A'$ passes through two diametrically opposite points of the sphere and thus its length is at least $2\pi$.

[1] Any continuous deformation of $A$ to a point must intersect $B$.

Shortly after this I found a published proof [2] based on the same idea as the first proof described above, but using the convex hull instead of the minimal surface.

# References


[1] W. Hayman, Meromorphic functions, Clarendon Press, Oxford, 1964.

[2] M. Edelstein and B. Schwatz, On the length of linked curves, Israel J. Math., 23, 1 1976, 94-95.