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On Deviations of Meromorphic Functions of Finite Lower Order

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For a function f meromorphic in the finite plane $\mathbf{C}$ let

$$\beta(a,f) = \varliminf_{r\to\infty} \ln M(r,(f-a)^{-1})/T(r,f), \qquad a \neq \infty,$$

and

$$\beta(\infty,f) = \varliminf_{r\to\infty} \ln M(r,f)/T(r,f).$$

Here and below we use the standard notation from the theory of meromorphic functions [1].

Several papers in recent years have dealt with the study of convergence for series $\sum_a \delta^\alpha(a,f)$, $\alpha < 1$, for meromorphic functions of finite lower order. The strongest result in this direction is due to Weitsman [2], who proved that if f is a meromorphic function of finite lower order, then

$$\sum_a \delta^{1/3}(a,f) < \infty. \tag{0.1}$$

Hayman ([1] §4.3) proved that the series $\sum_a \delta^{1/3-\varepsilon}(a,f)$ can diverge for any $\varepsilon > 0$. A detailed history of the problem is given in [2]. The quantities $\beta(a,f)$ (they are called the deviation values) were systematically studied by Petrenko [3], who proved, in particular, that

$$\sum_a \beta^{1/2}(a,f) \ln^{-1/2-\varepsilon} \frac{1}{\beta(a,f)} < \infty$$

for any $\varepsilon > 0$ for functions of finite lower order [4].

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In the present paper we prove the

THEOREM. *Let f be a meromorphic function of finite lower order. Then*

$$\sum_a \beta^{1/2}(a,f) < \infty. \tag{0.2}$$

This relation was conjectured in [3]. The theorem was announced by Barsegyan in Akad. Nauk Armyan. SSR Dokl. **67** (1978), no. 5. According to a letter from him, his proof contained a gap.

It is easy to deduce (0.1) from (0.2). Indeed, let

$$\theta(r,a) = \text{meas}\{\theta \in [0,2\pi] : \ln|f(re^{i\theta}) - a|^{-1} \geq \ln r\}, \qquad a \in \mathbf{C}.$$

For any finite colection $a_1, \dots, a_q \in \mathbf{C}$ the inequality $\sum_j \theta(r,a_j) \leq 2\pi$ holds for sufficiently large r. On the other hand, it is easy to see that

$$\delta(a_j,f) \leq \theta(r,a_j)\beta(a_j,f) + o(1), \qquad r \to \infty.$$

By Hölder's inequality,

$$\sum \delta^{1/3}(a,f) \leq (2\pi)^{1/3}\left(\sum \beta^{1/2}(a,f)\right)^{2/3}.$$

Therefore, (0.1) follows from (0.2).

Our theorem is sharp in the following sense. First, examples are known of meromorphic function of infinite lower order such that $\beta(a,f) > 0$ for uncountable sets of numbers $a \in \mathbf{C}$ ([3], p. 82). Second, analysis of known examples ([3], p. 47) shows that for any sequence (η_n) of numbers with $\eta_n > 0$ and $\sum_1^\infty \eta_n = 1$ there is a meromorphic function of normal type of order 1 such that $\beta(a_n,f) \geq \eta_n^2/4$, where $(a_n) \subset \mathbf{C}$ is a previously specified sequence.

Needed auxiliary results are given in §1, and the theorem is proved in §2.

1. LEMMA 1. *Let f be a meromorphic function, and let $a \in \mathbf{C}$. Then*

$$\log^+\left|\frac{f'(z)}{f(z)-a}\right| = o(T(12|z|,f)), \qquad z \to \infty,\ |z| \notin I,$$

where $I \subset (0,\infty)$ is such that $\text{meas}(I \cap (0,r)) = o(r)$, $r \to \infty$.

This lemma is a variant of Lemma 1.4.1 in [3]. We omit the simple proof, which is based on differentiation of the Schwarz-Jensen formula.

Let $D(R) = \{z : |z| < R\}$, and let $u \geq 0$ be the difference of two functions subharmonic on $D(R)$ and continuous on $\overline{D}(R)$. Such functions will be called admissible in what follows. The generalized Laplacian Δu is a signed measure with Jordan decomposition $\mu_u^+ - \mu_u^-$. We use the notation

$$M(r,u) = \max_\theta u(re^{i\theta}), \qquad n(r,u) = \mu_u^-(D(R)),$$

$$N(r,u) = \int_0^r n(t,u)\frac{dt}{t}, \qquad 0 \leq r \leq R.$$

For an admissible function u we consider the open set $D = \{z \in D(R) : u(z) > 0\}$. Let $g(z,\varsigma,D)$ denote the function defined as follows. If z lies in the same component of D as ς, then $g(z,\varsigma,D)$ is the (positive) Green's function of this component with pole at ς. In all the remaining cases $g(z,\varsigma,D)=0$. Denote by $u(\cdot,D)$ a function harmonic in D with $u(z,D)=u(z)$, $z\in\overline{D}(R)\backslash D$. The Riesz representation

$$u(z)=u(z,D)+\int_D g(z,\varsigma,D)d(\mu_u^- - \mu_u^+) \tag{1.1}$$

is valid for an admissible function u in $\overline{D}(R)$.

Denote by D^* the circular symmetrization of the open set D, i.e., the open set such that

$$\operatorname{meas}\{D\cap\{z\colon|z|=r\}\}=\operatorname{meas}\{D^*\cap\{z\colon|z|=r\}\},$$

where $D^*\cap\{z\colon|z|=r\}$ is either the whole circle or an arc whose midpoint lies on the positive ray. For any measurable function φ on $[-\pi,\pi]$ define the symmetrization φ^* as the monotonically decreasing function of $|\theta|$, $\theta\in[-\pi,\pi]$, such that

$$\operatorname{meas}\{\theta\colon\varphi(\theta)>t\}=\operatorname{meas}\{\theta\colon\varphi^*(\theta)>t\}$$

for any $t\in\mathbf{R}$. Let $u^*(\cdot,D^*)$ be defined as follows: $u^*(\cdot,D^*)$ is harmonic on D^* and equal to 0 on $D(R)\backslash D^*$, and $u^*(Re^{i\theta},D^*)=(u(Re^{i\theta}))^*$. The function $u^*(\cdot,D^*)$ is admissible.

LEMMA 2. 1°. *If u is an admissible function on $\overline{D}(R)$, $D=\{z\in D(R)\colon u(z)>0\}$, then*

$$M(r,u(\cdot,D))\geq M(r,u^*(\cdot,D^*))=u^*(r,D^*),\qquad 0\leq r\leq R.$$

2°. $M(r,g(\cdot,\varsigma,D))\leq M(r,g(\cdot,|\varsigma|,D^*))=g(r,|\varsigma|,D^+)$, $0\leq r\leq R$.

Assertion 1° follows from Theorem 7 of Baernstein in [5], and 2° is Theorem 5 in the same paper.

Denote by $c(\mu)$ the circular projection of the measure μ on the positive ray. Lemma 2 gives us

LEMMA 3. *Suppose that the admissible function u has the form* (1.1). *Then*

$$M(r,u)\leq M(r,u^*)=u^*(r),\quad r\leq R,\qquad n(r,u)=n(r,u^*),\quad r<R, \tag{1.2}$$

where

$$u^*(z)=u^*(z,D^+)+\int_{D^*}g(z,\varsigma,D^*)dc(\mu_u^-).$$

Note that $u^*=0$ in $D(R)\backslash D^*$, and that u^* is superharmonic in D^* and subharmonic off the positive ray.

LEMMA 4. *Suppose that u is an admissible function,*

$$\varlimsup_{|z|\to R}u(z)\leq 1, \tag{1.3}$$

$$\mu_u^-(D)<\infty. \tag{1.4}$$

Let $v(z)=\min\{u(z),2\}$. Then $n(R,v)=n(R,u)$.

PROOF. Let $D_1 = \{z \in D: u(z) > 2\}$. By (1.3) and the fact that $u(z) = 0$ for $z \in \partial D \cap D(R)$, we have that $\overline{D}_1 \subset D$. Let D_2 be an open set with smooth boundary Γ such that $\overline{D}_1 \subset D_2$ and $\overline{D}_2 \subset D$. Obviously, $u(z) = v(z)$ in a neigborhood of Γ. By Green's theorem,

$$\mu_v^-(D_2) = \int_\Gamma \frac{\partial v}{\partial n} ds = \mu_u^-(D_2) \tag{1.5}$$

in $D(R)\backslash D_1$ we have that $u(z) = v(z)$; therefore

$$\mu_v^-(D(R)\backslash D_2) = \mu_u^-(D(R)\backslash D_2),$$

which together with (1.5) proves the lemma.

LEMMA 5. *Let (v_k) be a sequence of admissible functions with the properties that*

$$n(R, v_k) \le A \quad (A \text{ does not depend on } k), \tag{1.6}$$
$$v_k(r) \ge \kappa > 0, \qquad R/8 \le r \le R,\ r \notin X_k, \tag{1.7}$$

$\operatorname{meas} X_k \to 0$, $k \to \infty$, *and κ does not depend on k. Then $M(r, v_k) \ge \kappa/2$, $R/4 \le r \le R/2$, for sufficiently large k.*

PROOF. We prove the lemma by contradiction. Suppose that (1.6) and (1.7) hold, and that there is a sequence $r_k \in [R/4, R/2]$ such that

$$M(r_k, v_k) < \kappa/2. \tag{1.8}$$

Consider the new sequence of functions

$$w_k(z) = \frac{2}{\kappa}\left(v_k\left(\frac{r_k}{2}z\right) - \frac{\kappa}{2}\right)^+.$$

It follows from (1.8), (1.7), and (1.6) that

$$w_k(2e^{i\theta}) = 0, \tag{1.9}$$
$$w_k(r) \ge 1, \quad 1 \le r \le 2,\ r \notin Y_k, \qquad \operatorname{meas} Y_k \to 0, \quad k \to \infty, \tag{1.10}$$
$$n(3, w_k) \le A. \tag{1.11}$$

Without loss of generality it can be assumed that the w_k are harmonic in $G = D(2)\backslash[1, 2]$. Indeed, if we replace w_k in G by the solution to the Dirichlet problem with boundary data w_k, then the conditions (1.9)–(1.11) are not violated. We next assume that the w_k are harmonic in G. Denote by $\omega(z, \alpha)$ the harmonic measure of an arc $\alpha \subset \partial G$ in G. For any continuous function u let

$$E(r, u) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\, d\theta.$$

It follows from (1.10) that

$$w_k(z) \ge \omega(z, ([1, 2]\backslash Y_k)) = \omega(z, [1, 2]) - \omega(z, Y_k) = \omega_1(z) - \omega_{2,k}(z), \ |z| < 2.$$

For any $\varepsilon > 0$ we have that $\omega_{2,k}(z) \to 0$ as $k \to \infty$ uniformly with respect to z for $\varepsilon \le |\arg| \le \pi$. Therefore, $E(r, \omega_{2,k}) \to 0$ uniformly for $0 \le r \le 2$. Consequently,

$$E(r, w_k) \ge E(r, \omega_1) + o(1), \qquad k \to \infty \tag{1.12}$$

uniformly with respect to r. For $\omega_1(z)$ there is an explicit expression

$$\omega_1(z) = \frac{2}{\pi}\arcsin\frac{2-|\varsigma|}{|2-\varsigma|}, \qquad \varsigma = \frac{4(z-1)}{4-z}.$$

From this an uncomplicated direct computation gives us that

$$\lim_{r\to 2-0} \frac{rd}{dr} E(r, \omega_1) = -\infty. \tag{1.13}$$

It follows from (1.9) that $E(2, w_k) = 0$. Considering (1.12) and (1.13), we get that

$$\lim_{k\to\infty} \lim_{r\to 2-0} \frac{rd}{dr} E(r, w_k) = -\infty.$$

However, by Green's formula, we get from (1.11) that

$$\begin{aligned}\frac{rd}{dr}E(r, w_k) &= n(r, -w_k) - n(r, w_k) \ge -n(r, w_k)\\ &\ge -n(3, w_k) \ge -A\end{aligned}$$

for $r < 2$. This contradiction proves the lemma.

Let Γ_1 and Γ_2 be two simple Jordan curves joining the circles of the annulus $\{z\colon 1 < |z| < 2\}$. Denote by S one of the curvilinear quadrangles bounded by these curves and arcs of the circles of the annulus. There is a unique conformal and univalent mapping of the domain S onto some rectangle $Q = \{\varsigma = \xi + i\eta\colon |\xi| < 2,\ |\eta| < \delta\}$ with the curves Γ_1 and Γ_2 going into the horizontal sides $|\eta| = \pm\delta$, and with the circular arcs going into the vertical sides.

LEMMA 6. *$\delta \le 2|S| \le 6\pi$, where $|S|$ is the area of the region S.*

This is a variant of a known theorem of Grötzsch.

PROOF. Let $\varphi\colon Q \to S$ be the mapping function. Then

$$1 \le \int_{-2}^{2} |\varphi'|\, d\xi, \quad 1 \le 4\int_{-2}^{2} |\varphi'|^2\, d\xi, \quad 2\delta \le 4\int_{-2}^{2} |\varphi'|^2\, d\xi\, d\eta = 4|S|$$

as required.

Let $w(\varsigma)$ be a superharmonic function continuous on $\overline{Q}$ such that $0 \le w(\pm 2 + i\eta) \le 2$ for $|\eta| \le \delta$, and $w(\xi \pm i\delta) = 0$ for $|\xi| < 2$ and $\delta < 6\pi$.

LEMMA 7. *Let $M(\xi) = \max_\eta w(\xi + i\eta) \ge \kappa > 0$, $|\xi| < 1$. Then $\kappa \le A(\delta\mu_w^-(Q) + \delta^2)$, where A is an absolute constant.*

PROOF. We represent w as the sum of a harmonic function h on Q and a Green's potential p. If $|\xi| \le 1$, then it is not hard to get the estimate

$$h(\varsigma) \le A_1 \exp(-A_2/\delta) \le A_3\delta^2, \qquad \operatorname{Re}\varsigma = \xi, \tag{1.14}$$

where A_1, A_2, and A_3 are absolute constants.

For the potential we have that

$$p(\varsigma) = \int_Q g(\varsigma, t, Q)\, d\mu_w^- \le \int_Q g(\xi, \operatorname{Re} t, Q)\, d\mu_w^-,$$

where g is the Green's function. Denote by $\Pi(\delta)$ the horizontal strip $\{\varsigma : |\operatorname{Im}\varsigma| < \delta\}$ and let $M_1(\xi) = \max_\eta p(\xi + i\eta)$. We have that

$$\begin{aligned} \int_{-1}^{1} M_1(\xi)\, d\xi &\le \int_{-1}^{1} d\xi \int_Q g(\xi, \operatorname{Re} t, \Pi(\delta))\, d\mu_w^- \\ &\le \int_Q d\mu_w^- \int_{-\infty}^{\infty} g(\xi, 0, \Pi(\delta))\, d\xi \\ &= \delta\mu_w^-(Q) \int_{-\infty}^{\infty} g(\xi, 0, \Pi(1))\, d\xi, \end{aligned} \tag{1.15}$$

because $g(\delta\xi, 0, \Pi(1)) = g(\xi, 0, \Pi(\delta))$. The last integral in (1.15) obviously converges and is an absolute constant. By (1.14) and (1.15),

$$\kappa \le \int_{-1}^{1} M(\xi)\, d\xi \le A_3\delta^2 + \int_{-1}^{1} M_1(\xi)\, d\xi \le A(\delta\mu_w^-(Q) + \delta^2),$$

which is what we were required to prove.

2. Proof of Theorem. Without loss of generality it can be assumed that $f(0) = 1$ and that $\overline{N}(r, f) \sim T(r, f)$, $r \to \infty$; consequently,

$$2T(r, f) \le T(r, f') \le 2T(2r, f) = 2T(2r).$$

It is known ([3], p. 64) that for functions of finite lower order the series $\sum_a \beta(a, f)$ converges; therefore, for the proof it can be assumed that the numbers $\beta(a) = \beta(a, f)$ are sufficiently small. Further, if the lower order λ of f is equal to 0, then the relation $\beta(a, f) > 0$ can hold for at most one value $a \in \mathbf{C}$ ([3], p. 69). Therefore, it will be assumed that $\lambda > 0$.

There exist sequences $r_m \to \infty$ and $S_m \to \infty$ such that

$$T(Sr_m) \le S^{\lambda+1} T(r_m), \qquad 1 \le S \le S_m. \tag{2.1}$$

The proof of the theorem is divided into several steps.

1°. By H. Cartan's theorem ([1], Theorem 1.3) and by (2.1),

$$\begin{aligned} \int_0^{2\pi} n\left(4r_m, \frac{1}{f' - te^{i\varphi}}\right) d\varphi &\le \int_0^{2\pi} N\left(12r_m, \frac{1}{f'/t - e^{i\varphi}}\right) d\varphi + \text{const} \\ &\le \log + \frac{1}{t} + (2 + o(1))T(24r_m) \\ &\le A\left(\log^+ \frac{1}{t} + T(r_m)\right) \quad \forall t > 0. \end{aligned}$$

Here and below, A denotes various constants depending only on λ. Let $l(t)$ be the total length of the level curves $|f'(z)| = t$ in the disk $D(4r_m)$, and let $\gamma_1 = \exp(-T(r_m))$ and $\gamma_2 = \gamma_1/2$. According to the length and area principle ([6], §2.1),

$$\int_{\gamma_2}^{\gamma_1} \frac{l^2(t)dt}{t} \le Ar_m^2 \left(\log^+ \frac{1}{\gamma_2} + T(r_m)\right).$$

Therefore, there is an α_m, $\sqrt{T(r_m)} \le \alpha_m \le \sqrt{T(r_m)} + \log 2$, such that

$$l(e^{-\alpha_m}) \le Ar_m\sqrt{T(r_m)}. \tag{2.2}$$

Fix a finite collection of points $a_1, \dots, a_q \in \mathbf{C}$, $q \ge 2$, with $\min\{|a_i - a_j| : i \ne j\} = c > 0$ and $\beta(a_j, f) > 0$. We consider the set $G_m = \{z : |z| < 4r_m, \log|f'(z)| < -\alpha_m\}$. Let G_{jm}, $1 \le j \le q$, be the open set formed by the components G_m containing a point z_1 at which

$$|f(z_1) - a_j| < c/4. \tag{2.3}$$

Then

$$|f(z) - a_j| < c/2 \tag{2.4}$$

everywhere in G_{jm}. Indeed, $z \in G_{jm}$. In the same component as z there is a point z_1 for which (2.3) holds. By (2.2), there is a curve $\Gamma \subset G_{jm}$ joining z and z_1 with length at most $Ar_m\sqrt{T(r_m)}$. On this curve, as everywhere in G_m,

$$|f'(z)| \le \exp(-\alpha_m) \le \exp(-\sqrt{T(r_m)}).$$

Therefore, considering that $\lambda > 0$, we get that

$$|f(z) - f(z_1)| \le \int_\Gamma |f'(z)|\,|dz| \le A\exp(-\sqrt{T(r_m)})r_m\sqrt{T(r_m)} = o(1), \quad m \to \infty,$$

and (2.3) implies (2.4). In particular, the G_{jm} are pairwise disjoint.

2°. By Lemma 1 and (2.1), there is a set $I_m \subset [r_m/2, 4r_m]$, meas $I_m = o(r_m)$, $m \to \infty$, such that

$$\log^+\left|\frac{f'(z)}{f(z) - a_j}\right| = o(T(r_m)), \qquad m \to \infty, \tag{2.5}$$

$1 \le j \le q$, $|z| \in [r_m/2, 4r_m]\backslash I_m$. We show that for any $r \in [r_m/2, 4r_m]\backslash I_m$ there is a point z with $|z| = r$ such that $z \in G_{jm}$ and

$$\log|f'(z)| < -A\beta(a_j)T(r_m). \tag{2.6}$$

Indeed, since $\beta(a_j) > 0$, for any $r \in [r_m/2, 4r_m]$ there is a point z with $|z| = r$ such that

$$\log|f(z) - a_j| < -\tfrac{1}{2}\beta(a_j)T(r) \le -A\beta(a_j)T(r_m). \tag{2.7}$$

This and (2.5) give us (2.6) for some point z. By the definition of G_m, this point is contained in G_m. Finally, by (2.4), it is contained precisely in G_{jm}.

We remark that G_{jm} cannot contain any circle $\{z : |z| = r\}$. This fact (which is important for what follows) is a consequence of (2.4), (2.5) and the fact that $q > 2$.

3°. By a theorem of Miles [7], the meromorphic function $1/f'$ can be represented as a quotient of two entire functions g_1 and g_2 such that $T(r, g_j) \le A_1T(A_2r)$, $j = 1, 2$, where A_1 and A_2 are absolute constants. Using this theorem and the known estimate of the maximal modulus of an entire function in terms of the characteristic, we get that $-\log|f'(z)| = t_1(z) - t_2(z)$, where t_1 and t_2 are subharmonic functions with

$$t_j(z) \le AT(r_m), \qquad |z| < 12r_m. \tag{2.8}$$

Let

$$t_1^* = \max(t_1, t_2 + \alpha_m), \quad t_2^* = \max(t_1 - T(r_m), \quad t_2 + \alpha_m),$$

$$y_m(z) = (T(r_m)^{-1}) \times (t_1^*(r_m z) - t_2^*(r_m z)), \qquad z \in D(4).$$

Denote by D_{jm}, D_m, and X_m the sets such that

$$r_m D_{jm} = G_{jm}, \quad r_m D_m = G_m, \quad r_m X_m = I_m.$$

It is not hard to see that $y_m(z) = 0$ if $z \in D(4) - D_m$, $y_m(z) = 1$ if $\log|f'(r_m z)| \le -T(r_m) - \alpha_m$, and $y_m(z) = T(r_m)^{-1}(-\log|f'(r_m z)| - \alpha_m)$ otherwise. Therefore

$$0 \le y_m \le 1, \qquad z \in \overline{D}(4), \tag{2.9}$$

and it follows from (2.6) that

$$\max_{|z|=r, z\in D_{jm}} y_m(z) \ge A\beta(a_j), \qquad r \notin X_m,\ 1/2 \le r \le 4. \tag{2.10}$$

Here and below, it is assumed in analogous inequalities that $A\beta(a_j) < 1$. By (2.8),

$$\begin{aligned} n(4, y_m) &= T(r_m)^{-1} n(4r_m, -t_2^*) \le (Tr_m)^{-1} N(12r_m, -t_2^*) \\ &\ge T(r_m)^{-1} M(12r_m, t_2^*) \le A, \end{aligned} \tag{2.11}$$

where A does not depend on m.

Now let

$$u_{jm} = \begin{cases} y_m(z), & z \in D_{jm}, \\ 0, & z \in D(4)\backslash D_{jm}. \end{cases}$$

It follows from (2.10) that

$$M(r, u_{jm}) \ge A\beta(a_j), \qquad \tfrac{1}{2} \le r \le 4,\ r \notin X_m, \tag{2.12}$$

and it follows from (2.9) that

$$0 \le u_{jm} \le 1, \qquad z \in \overline{D}(4). \tag{2.13}$$

Let $p_{jm} = n(4, u_{jm})$. We get from (2.11) that

$$\sum_{j=1}^{q} p_{jm} \le n(4, y_m) \le A. \tag{2.14}$$

The functions u_{jm} satisfy the conditions of Lemma 3 (take D_{jm} as D and $R = 4$). According to this lemma, we get functions u_{jm}^* and regions D_{jm}^* satisfying the following conditions:

$$M(r, u_{jm}^*) \ge A\beta(a_j), \qquad 1/2 \le r \le 4,\ r \notin X_m; \tag{2.15}$$

$$n(4, u_{jm}^*) \le p_{jm} \le A; \tag{2.16}$$

$$\overline{\lim_{|z|\to 4}}\, u_{jm}^*(z) \le 1; \tag{2.17}$$

$$u_{jm}^*(z) = 0, \qquad z \in D(4)\backslash D_{jm}^*. \tag{2.18}$$

Inequality (2.15) follows from (2.12) and Lemma 3; (2.16) follows from (1.2); (2.17) follows from (2.13); and (2.18) follows from the remark after Lemma 3. By (2.15) and (2.16), the conditions of Lemma 5 hold ($R = 4$), and this lemma enables us to replace (2.15) by

$$M(r, u^*_{jm}) \geq A\beta(a_j), \qquad 1 \leq r \leq 2. \tag{2.19}$$

Let us now consider the functions $v_{jm} = \min(u^*_{jm}, 2)$. We get from (2.19) that

$$M(r, v_{jm}) \geq A\beta(a_j), \qquad 1 \leq r \leq 2. \tag{2.20}$$

By (2.16)–(2.18), conditions (1.3) and (1.4) of Lemma 4 are satisfied. This lemma gives us that

$$n(4, v_{jm}) \leq p_{jm}. \tag{2.21}$$

We now observe that since the regions D_{jm} are disjoint,

$$\sum_{j=1}^{q} |D^*_{jm}| = \sum_{j=1}^{q} |D_{jm}| \leq 16\pi. \tag{2.22}$$

Further, none of the D_{jm} contains a circle about zero, as follows from the remark at the end of 2°. Therefore, the regions D^*_{jm} also do not contain such circles. It follows from (2.20) and (2.18) that $[1, 2] \subset D^*_{jm}$; consequently, the sets $S_{jm} = D^*_{jm} \cap \{z: 1 < |z| < 2\}$ are connected. It is easy to see that the S_{jm} are simply connected domains.

We map each domain S_{jm} conformally and univalently onto the rectangle $Q_{jm} = \{\varsigma = \xi + i\eta : |\xi| < 2; |\eta| < \delta_{jm}\}$ as required in Lemma 6. According to this lemma,

$$\delta_{jm} \leq 2|S_{jm}| \leq 2|D^*_{jm}|. \tag{2.23}$$

Let $\varphi_{jm}: Q_{jm} \to S_{jm}$ be the conformal univalent mapping inverse to the indicated mapping, and consider the composition $w_{jm}(\varsigma) = v_{jm}(\varphi_{jm}(\varsigma))$. By the definition of v_{jm}, it follows that $0 \leq w_{jm} \leq 2$, and $w_{jm}(\xi + i\delta_{jm}) = 0$; by (2.21),

$$\mu^-_{w_{jm}}(Q_{jm} \leq p_{jm}, \tag{2.24}$$

and, by (2.20),

$$\max w_{jm}(\xi + i\eta) \geq A\beta(a_j), \qquad |\xi| < 2.$$

Lemma 7 (with $\kappa = A\beta(a_j)$) together with (2.24) and (2.23) gives us that

$$\beta(a_j) \leq A(\delta_{jm} p_{jm} + \delta^2_{jm}) \leq 4A(|D^*_{jm}| p_{jm} + |D^*_{jm}|^2).$$

From this, using (2.14), (2.22), and elementary inequalities, we deduce that

$$\sum_{j=1}^{q} \beta^{1/2}(a_j) \leq A \sum_{j=1}^{q} |D_{jm}| + \sum_{j=1}^{q} p_{jm} \leq A.$$

The theorem is proved.

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