Green's formula and Cauchy theorem

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Consider a region in the plane

$$D = \{(x, y) : a < x < b,\ \phi_1(x) < y < \phi_2(x)\}, \quad (1)$$

where $\phi_1 < \phi_2$ are two continuous piecewise smooth functions.

Let u be a continuously differentiable function in the closure of this region. Applying the Newton–Leibniz formula, we obtain

$$\int_{\phi_1(x)}^{\phi_2(x)} \frac{\partial u}{\partial y} dy = u(\phi_2(x)) - u(\phi_1(x)), \quad a < x < b.$$

Now integrate this with respect to x from a to b, and obtain

$$\int\int_D \frac{\partial u}{\partial y} dxdy = \int_a^b u(\phi_2(x))dx - \int_a^b u(\phi_1(x))dx. \quad (2)$$

Definition 1. Let D be a region whose boundary consists of piecewise smooth curves. The *oriented boundary* of D is a parametrization of the boundary such that the region stays on the left when the point moves in the direction of increase of the parameter.

When using notation ∂D for a collection of curves, we always mean the oriented boundary.

For example, for the region D defined in (1) the parametrized boundary consists of 4 curves:

$$\gamma_1(t) = t + i\phi_1(t), \quad a \leq t \leq b,$$

$$\gamma_2(t) = b - t + i\phi_2(b - t), \quad 0 \leq t \leq b - a,$$

and two vertical segments. Since on the vertical segments we have $dx = 0$, on γ_1 we have $dx = dt$ and on γ_2, $dx = -dt$, formula (2) can be rewritten as

$$-\int\int_D \frac{\partial u}{\partial y} dxdy = \int_{\partial D} u\, dx. \tag{3}$$

Now, suppose that we have a region of the form

$$D = \{(x,y) : a < y < b,\ \phi_1(y) < x < \phi_2(y)\}, \tag{4}$$

and a continuously differentiable function v in it. Applying the same argument, we obtain

$$\int\int_D \frac{\partial v}{\partial x} = \int_{\partial D} u\, dy. \tag{5}$$

Make a picture, and see why (3) has a minus sign, and (5) does not!

If a region can be described by each of the formulas (1) and (4), for example, if it is a rectangle or a triangle, then both formulas (3) and (5) apply, so they can be combined, and we obtain

$$\int_{\partial D} udx + vdy = \int\int_D \left(-\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) dxdy. \tag{6}$$

This is the Green formula. An expression of the type

$$udx + vdy = u(x,y)dx + v(x,y)dy$$

is called a *differential form*, or, more precisely a differential 1-form.

We proved (6) under the condition that D has both representations (1) and (4), but this condition is easily disposed of by the following argument. Suppose we have some region D (whose boundary consists of piecewise smooth curves), and let us cut it into two non-overlapping pieces D_1 and D_2 by some piecewise smooth curve γ in D. Then for the double integrals of any function u we have

$$\int\int_D u\ dxdy = \int\int_{D_1} u\ dxdy + \int\int_{D_2} u\ dxdy.$$

But we also have a similar formula for the line integrals:

$$\int_{\partial D} udx + vdy = \int_{\partial D_1} (udx + vdy) + \int_{\partial D_2} (udx + vdy).$$

This is an important statement, think about it and draw a picture! The contribution of the cut γ in the RHS *cancels*. This is the whole point of defining the oriented boundary as we did.

Now, it is intuitively evident that every reasonable region (whose boundary consists of piecewise smooth curves) can be broken by some cuts into "nice regions" to which both descriptions (1) and (4) apply, and this establishes the Green formula (6) for arbitrary region bounded by piecewise smooth curves.

We do not prove here this geometric statement about possibility of breaking the region rigorously. In all examples in this course it will be evident how to break the region.

So formula (6) can be applied to any region bounded by finitely many smooth curves and any continuously differentiable functions u and v.

Example. Let us take $u(x, y) = y$ and $v = 0$. We obtain

$$\text{area } D = \int\int_D dxdy = -\int_{\partial D} ydx.$$

This means that the *area* of a region can be computed by integrating something over the *boundary* of this region. There is a simple mechanical device, which is called planimeter, which does this: you trace the boundary of a region with this device, and it computes the integral giving you the area.

It is remarkable that certain double integrals over a region can be computed only from the data on the boundary of this region. Green's formula can be considered as a 2-dimensional generalization of the Newton-Leibniz formula. Similar generalizations exist in any dimensions.

Definition 2. A differential form $u(x, y)dx + v(x, y)dy$ is called an *exact* if

$$u_y = v_x. \tag{7}$$

So for exact differential forms, the RHS in the Green formula (6) vanishes, so the LHS also vanishes and we obtain

$$\int_{\partial D} (udx + vdy) = 0,$$

whenever $udx + vdy$ is exact in some region containing $\overline{D}$.

Let $f(z) = U(x,y) + iV(x,y)$ be an analytic function, so that U and V satisfy the Cauchy–Riemann conditions:

$$U_x = V_y, \quad U_y = -V_x.$$

Then the differential form

$$\begin{aligned} f(z)dz &= (U+iV)(dx+idy) = (Udx - Vdy) + i(Vdx + Udy) \\ &= (U+iV)dx + i(-V+iU)dy \end{aligned}$$

is exact, in the sense that both real and imaginary parts are exact. This follows from the Cauchy–Riemann equations. Indeed,

$$(U+iV)_y = U_y + iV_y = -V_x + iU_x = (-V+iU)_x,$$

so $u = U + iV$ and $v = -V + iU$ satisfy the exactness condition (7). So we obtain

Cauchy's theorem. *Let f be an analytic function in a bounded closed region $\overline{D}$, whose boundary consists of piecewise smooth curves. Then*

$$\int_{\partial D} f(z)dz = 0.$$

This is *Cauchy's theorem*, the central theorem of our subject. I recall that $\overline{D} = D \cup \partial D$, and analytic in $\overline{D}$ means that f is analytic on some open set containing $\overline{D}$.

Sometimes another form of Cauchy's theorem is convenient. Suppose that we have two curves γ_0 and γ_1 in a region Ω, both defined on $[0,1]$ and having common ends, that is

$$\gamma_1(0) = \gamma_2(0), \quad \gamma_1(1) = \gamma_2(1).$$

Definition 3. A *deformation* of γ_0 to γ_1 in Ω is a continuous function γ from the rectangle

$$\{(s,t) : 0 \leq t \leq 1, \quad 0 \leq s \leq 1\}$$

to Ω such that

$$\gamma(0,t) = \gamma_0(t), \quad \gamma(1,t) = \gamma_1(t).$$

If such a deformation exists, we say that two curves can be deformed one to another in Ω. Of course, this depends not only on the curves themselves, but also on the region Ω. For example, any two curves sharing their ends in the plane can be deformed into each other. But in $\mathbf{C}^* = \{z \in \mathbf{C} : z \neq 0\}$, the upper half of the unit circle cannot be deformed into the lower half.

Cauchy theorem, second form. *If f is analytic in a region Ω, and γ_1, γ_2 are two curves with common ends which can be deformed one into another in Ω, then*

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz. \tag{8}$$

In particular, if a closed curve γ can be deformed into a point (=constant curve) in the region Ω where f is analytic, then

$$\int_{\gamma} f(z)dz = 0. \tag{9}$$

There are regions where any two curves with common ends can be deformed into each other. (It is equivalent to saying that every closed curve can be deformed to a point). Such regions are called *simply connected*. For example, any convex region is simply connected. This can be generalized as follows: a region is called *starlike*, if it contains a point a such that every segment $[a, z]$, where z is in the region, is contained in the region. Every starlike region is simply connected. For example, the plane with the negative semi-axis removed.

In a simply connected region, (9) holds for every closed curve, and (8) holds for any two curves with common ends.