What is the true exceptional set in Gross' Theorem?

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Let f be a meromorphic function in the complex plane. Suppose that z is not a critical point, $w = f(z)$, and consider the germ ϕ_z of the inverse, such that $\phi_z(w) = z$. Gross' Theorem [1] says that ϕ_z has an analytic continuation along almost every ray $\{w + re^{i\theta} : 0 \leq r < \infty\}$ (with respect to the Lebesgue measure on the unit circle).

Can the exceptional set be improved in this theorem?

The only known example, where the exceptional set has the power of continuum, is given by Theorem 17 on p. 71 of [2]. In this example the exceptional set has zero capacity, and it is in fact much smaller than that.

[1] R. Nevanlinna, Analytic Functions.
[2] L.I. Volkovyskii, Research on the type problem of a simply connected Riemann surface, Proc. Steklov Inst. Math., XXXIV, Moscow 1950. (Russian).