Harmonic functions

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March 18, 2024

A C^2 function in a region D is called *harmonic* if it satisfies the *Laplace equation*

$$\Delta u := u_{xx} + u_{yy} = 0.$$

Laplace equation makes sense in any dimension:

$$\Delta u := u_{x_1,x_1} + u_{x_2,x_2} + \ldots + u_{x_n,x_n} = 0,$$

and it plays an important role in physics. One reason of this is that the function

$$|x|^{2-n} = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1-n/2}$$

is harmonic (check this!) in $\mathbf{R}^n \backslash \{0\}$. When $n = 3$ this function describes electrostatic potential (Coulomb's Law) and gravitational potential (Newton's gravitation law) and many other things. In dimension 2 a similar role is played by the function $-\log|z|$, which is harmonic in $\mathbf{C}^*$.

We will be concerned only with real-valued harmonic functions in regions in the plane. The simplest physical interpretation is a stationary (time independent) temperature in a thin plate.

Theorem. *In any simply connected region in the plane, every harmonic function is the real part of an analytic function f. This f is defined up to addition of a pure imaginary constant.*

Proof. Let u be a harmonic function. Consider the complex function

$$f = u_x - iu_y.$$

It is analytic since it satisfies the Cauchy–Riemann conditions (verify this!). Since the domain is assumed to be simply connected, there exists a primitive

$$F = U + iV.$$

We have $F' = f$, so F is analytic, and $U_x = u_x$ and $U_y = -V_x = u_y$, therefore $U - u =: c$ is constant. So $\mathrm{Re}\,(F - c) = u$.

The assumption that the region is simply connected is essential:

Exercise. *Function $u(z) = \log|z|$ is harmonic in $\mathbf{C}^*$, but it is not the real part of any analytic function in $\mathbf{C}^*$.*

We use the notation $D(a,r) = \{z : |z-a| < r\}$, $\overline{D}(a,r) = \{z : |z-a| \leq r\}$ and $\partial D(a,r) = \{z : |z-a| = r\}$ (oriented counterclockwise).

Average property. *If D is a region, and u is harmonic in D, then for every closed disk such that $\overline{D}(a,r) \subset D$,*

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{it})dt.$$

Physical interpretation: the temperature at any point of the plate equals to the average of the temperatures over the boundary of a disk centered at this point.

This follows from the similar property of analytic functions (which in turn is the consequence of Cauchy's integral formula) by taking the real parts.

As a corollary we obtain the further important properties

Maximum/Minimum Principle *If a harmonic function has a (non-strict, local) maximum or minimum at some point of the region of harmonicity, then this function is constant.*

Uniqueness Property *If two functions are harmonic in a region D with compact closure, continuous in $\overline{D}$ and coincide on ∂D, then they coincide everywhere in D.*

Indeed $\overline{D}$ is compact, so if we have two continuous functions in $\overline{D}$, then their difference must have a maximum and minimum in $\overline{D}$. By the Maximum/Minimum Principle, they must be attained on ∂D, but if the functions are equal on ∂D, these maximum and minimum are both 0.

The principal problem about harmonic functions is called the

Dirichlet Problem. *Let D be a region, and a function ϕ is given on ∂D.*

Find a harmonic function u in D which takes the boundary values ϕ, that is

$$\lim_{z\to\zeta} u(z) = \phi(\zeta), \quad \textit{for every} \quad \zeta \in \partial D. \tag{1}$$

Uniqueness property shows that for a continuous function ϕ and bounded region D the Dirichlet problem has at most one solution.

We want to slightly generalize this fact, to deal with unbounded regions, and some discontinuous functions.

First we notice that Maximum Principle can be restated as follows:

Proposition. *Let u be a harmonic function in a region D with compact closure, and suppose that*

$$\limsup_{z\to\zeta} u(z) \leq 0, \quad \textit{for all} \quad \zeta \in \partial D.$$

Then $u(z) \leq 0$ for all z in D.

Phragmén–Lindelöf Principle. *Let $D \subset \mathbf{C}$ be a bounded region, and u is a* **bounded** *harmonic function in D. Let a be a point on the boudary of D, and suppose that*

$$\limsup_{z\to\zeta} u(z) \leq 0 \quad \textit{for all} \quad \zeta \in \partial D \backslash \{a\}. \tag{2}$$

Then $u \leq 0$ in D.

Proof. Without loss of generality we may assume that D is contained in the unit disk and $a = 0$. For every $\epsilon > 0$ consider the function

$$u_\epsilon(z) = u(z) + \epsilon \log |z|.$$

Then u_ϵ satisfies the assumptions of the Proposition above, and thus $u_\epsilon(z) \leq 0$ for all $z \in D$. Fixing ariitrary $z \in D$, pass to the limit when $\epsilon \to 0$, and we obtain the required conclusion.

With a similar proof, one can relax the assumption, and require that (2) holds for all boundary points of D, except finitely many. Then the same conclusion holds.

We can also relax the condition that D is bounded: it is sufficient to require that there is a conformal map of D onto a bounded region. For example, the Phragmén–Lindelöf Principle applies to a half-plane.

Out next task is to solve Dirichlet's problem for simple regions, namely disks and half-planes. We begin with the upper half-plane which we denote by H.

For every real t, the function $u(z) = \text{Arg}\,(z-t)$ is harmonic and bounded in the upper half-plane and has these boundary values:

$$u(x) = \begin{cases} 0, & x > t, \\ \pi, & x > t. \end{cases}$$

The limit as $z \to t$ does not exist. So, for $t_1 < t_2$,

$$u_{t_1,t_2}(z) = \frac{1}{\pi}\left(\text{Arg}\,(z-t_2) - \text{Arg}\,(z-t_1)\right)$$

tends to the characteristic function of the interval (t_1, t_2) as z tends to the real line. If $z = x+iy$, we can rewrite this function as

$$\frac{1}{\pi}\left(\arctan\frac{y}{x-t_2} - \arctan\frac{y}{x-t_1}\right) = \frac{y}{\pi}\int_{t_1}^{t_2}\frac{dt}{(x-t)^2+y^2}.$$

Now, by taking a linear combination of such functions, we can solve the Dirichlet problem for any step function ϕ on the real line (of course, (1) will hold only at the points of continuity of ϕ). Then, since every continuous function ϕ with bounded support can be uniformly approximated by step functions, we obtain the solution of Dirichlet's problem for such functions ϕ in the form

$$u(x+iy) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\phi(t)dt}{(x-t)^2+y^2}. \tag{3}$$

The RHS is called the Poisson integral (for the upper half-plane). It is the convolution of ϕ with the *Poisson kernel*

$$P(y,t) = \frac{y}{\pi(t^2+y^2)}.$$

Poisson's kernel has these three important properties:

a) $P(y,t) > 0$ for $y > 0$ and $t \in \mathbf{R}$.

b) $\int_{\mathbf{R}} P(y,t)dt = 1$ for all $y > 0$.

c) For every $\delta > 0$,

$$\lim_{y\to 0+}\int_{|t|\geq\epsilon} P(y,t)dt = 0.$$

One-parametric families of functions of t (here the parameter is y) are called *positive kernels*. There is a general theorem about convolutions of continuous functions with positive kernels.

Theorem. *If $P(y,t)$ is a positive kernel, and ϕ a bounded continuous function then the convolution*

$$x \mapsto u(x,y) = \int_{-\infty}^{\infty} P(y,x-t)\phi(t)dt \tag{4}$$

tends to $\phi(x)$.

Proof. Using b) we write

$$\phi(x) = \int_{-\infty}^{\infty} P(y,x-t)\phi(x)dt. \tag{5}$$

Now continuity of ϕ means that for every $\epsilon > 0$ there exists $\delta > 0$ such that $|\phi(t) - \phi(x)| < \epsilon$ for $|x-t| < \delta$. Propsrties a) and c) imply that the contribution to both integrals (4) and (5) from $\{t : |x-t| \geq \delta\}$ tends to zero as $y \to 0$. So we have

$$|u(x,y) - \phi(x)| \leq \int_{|x-t|<\delta} P(y,x-t)|\phi(t) - \phi(x)|dt + o(1) \leq \epsilon + o(1).$$

This proves the theorem.

There are several ways to obtain an analogous result for the unit disk. For example, one can use a conformal map of the disk to the upper hapfplane, and using this map to make a change of the variable in the Poisson formula (3). Another method is used in Ahlfors' book. But it is simpler to find an appropriate positive kernel for the unit disk, and refer to the Theorem above.

Function

$$S(z) = \frac{1+z}{2\pi(1-z)}$$

maps the unit disk onto the upper half-plane. By splitting it to the real and imaginary parts we obtain

$$S(z) = P(z) + iQ(z) := \frac{1}{2\pi}\frac{1-|z|^2}{|1-z|^2} + i\frac{1}{\pi}\frac{\operatorname{Im} z}{|1-z|^2}.$$

We claim that $P(re^{it})$ is a positive kernel, where the parameter w is $r \in (0,1)$, that is

a') $P(z) > 0$ for $|z| < 1$,

b') $\int_{-\pi}^{\pi} P(re^{it})dt = 1$,

c') $\int_{\epsilon}^{2\pi-\epsilon} P(re^{it})dt \to 0$ as $r \to 1$.

Compare with a), b), c) above. Property a') is evident since $\mathrm{Re}\, S(z) > 0$ by definition. To evaluate the integral in b') we notice that such an integral of Q is zero (function $t \mapsto Q(re^{it})$ is odd), so it is enough to evaluate the integral of S.

$$\int_{-\pi}^{\pi} S(re^{it})dt = \frac{1}{\pi}\int_{|z|=r} \frac{1+z}{1-z}\frac{dz}{iz} = \frac{1}{2\pi} 2\pi i \,\mathrm{res}_{\,z=0} \frac{1+z}{iz(1-z)} = 1.$$

Finally property c') is evident from the explicit expression of P. Using the above Theorem we obtain the

Poisson's Formula for the unit disk. *Let ϕ be a continuous function on the unit circle, then the function*

$$u(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-|z|^2}{|1-ze^{-it}|^2}\phi(e^{it})dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(1-r^2)\phi(e^{it})dt}{1-2r\cos(t-\theta)+r^2}.$$

is harmonic in $|z| < 1$ and has boundary values $\phi(e^{it})$.

Now recall that an analytic function is defined by its real part, up to an additive imaginary constant. So we obtain

Schwarz's Formula for the unit disk. *If f is analytic in a region containind $|z| \leq 1$ then*

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1+ze^{-it}}{1-ze^{-it}}.\mathrm{Re}\, f(e^{it})dt + i\mathrm{Im}\, f(0).$$

Exercise. *Prove the Poisson formula for the disk of radius R:*

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(R^2-r^2)\phi(Re^{it})}{R^2-2Rr\cos(\theta-t)+r^2}.$$

and the Schwarz formula for the disk of radius R:

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{Re^{it} + z}{Re^{it} - z} \operatorname{Re} f(Re^{it}) dt + i \operatorname{Im} f(0).$$

An important consequence is the estimate of the modulus of f in terms of its real past. Let

$$M(r, f) = \max_{|z|=r} |f(z)|, \quad \text{and} \quad A(r, f) = \max_{|z|=r} |\operatorname{Re} f(z)|.$$

If f is analytic in $|z| \leq R$, and $r \in (0, R)$, then we have

$$M(r, f) \leq \frac{R + r}{R - r} A(r, f) + |f(0)|.$$

This is called the Hadamard's inequality.

Exercise. *Prove that if u is a harmonic function in $\mathbf{C}$ and*

$$u(z) = O(z^d), \quad z \to \infty$$

for some integer d, then u is the real part of a polynomial of degree at most d.

Removable singularity theorem. *Let u be a bounded harmonic function in $\{z : 0 < |z| < R\}$. Then u extends to a harmonic function in the whole disk.*

Proof. Choose $r \in (0, R)$ and let v be the solution of Dirichlet's problem for $|z| < r$ with boundary values $u(re^{it})$. Then $u - v$ is bounded, and zero on the boundary the ring $0 < |z| < r$. So by Phragmén–Lindelöf Principlle, it is zero identically.