Hayman's contribution to the theory of meromorphic functions

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Walter Hayman was a leader in classical complex analysis for several decades, and his 200+ publications, including 5 books, cover almost all aspects of this vast subject.
So for this talk I have to choose only one part of his great output, and based on my own scientific interests I choose the theory of meromorphic functions.
When I met my adviser, A. A. Goldberg, for the first time, he recommended me to read three books among which there was Hayman's Meromorphic functions[1]. I read it since then, and my opinion it is an unsurpassed example of mathematical exposition. This opinion is probably shared by many others: the book has 1666 citations on Mathscinet!

[1]The other two were Wittich and Pólya-Szego.

The book has a dedication:
"To Rolf Nevanlinna, creator of the modern theory of meromorphic functions"
This more or less defines the scope of my talk, so let me begin with recalling the main notions of Nevanlinna theory.
Let f be a meromorphic function, and $n(r,f)$ the number of poles in $|z| \leq r$, counting multiplicity. Then

$$N(r,f) := \int_0^r (n(t,f) - n(0,f))\frac{dt}{t} + n(0,f)\log r,$$

$$m(r,f) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+ |f(re^{i\theta}|d\theta,$$

where $a^+ = \max\{0,a\}$, and

$$T(r,f) := m(r,f) + N(r,f)$$

is the Nevanlinna characteristic.

A more intuitive definition is due to Ahlfors,

$$T_0(r,f) := \frac{1}{\pi}\int_{|t|<\leq r} \frac{|f'(te^{i\theta}|^2}{(1+|f(te^{i\theta})|^2)^2} tdtd\theta.$$

Since $T(r,f) = T_0(r,f) + O(1),\ r \to \infty$, we will not distinguish T and T_0. Then we define

$$m(r,a) = m(r,(f-a)^{-1}), \quad N(r,a) = N(r,(f-a)^{-1}),$$

and the First Main Theorem of Nevanlinna says that

$$m(r,a) + N(r,a) = T(r,f) + O(1).$$

The Nevanlinna characteristic is an analog of degree of a rational function in the transcendental case: for rational functions $T(r,f) = (d + o(1)) \log r$, while for transcendental functions we always have $\log r = o((T(r,f))$.

The Second Main Theorem says that for any distinct $a_1, \dots, a_q$ we have

$$\sum_{j=1}^{q} m(r, a_j) + N_1(r, f) \leq 2T(r, f) + S(r, f),$$

where N_1 counts the critical points of f, and $S(r, f)$ is the small "error term",

$$S(r, f) = O(\log rT(r, f)), \quad r \to \infty, r \notin E,$$

where E is a set of finite length.

The appearance of exceptional sets in inequalities is a characteristic feature of the theory (I think they appear for the first time in the theory of Wiman-Valiron), and a natural question can be asked whether they are really necessary.

This was answered by Hayman in 1972: *For every increasing sequence F_n of compact sets of zero capacity, and any two functions Φ_1, Φ_2 tending to $+\infty$, there exists an entire function f and a sequence $r_n \to \infty$ such that*

$$N(r_n, a, f) \leq \Phi_1(r_n), \quad a \in F_n$$

and

$$T(r_n, f) \geq \Phi_2(r_m).$$

This was the first paper of Hayman that I read. It is interesting to read Andre Bloch's speculations in his "La conception actuelle de la théorie de fonctions entières et méromorphes (L'Ens. Math., 1926) where he predicts such example from his philosophical principles. One can even see in his speculations an anticipation of Hayman's method of constructing this example, which consists of approximation by f in a sequence of rings of a function in the unit disk, the universal covering of the complement of F_n.
From conversations with Walter I know that he never read this paper of Bloch.

The main subject of this paper of Hayman is not the exceptional set in the Second Main theorem but the set of Valiron deficiencies:

$$\Delta(a, f) = \limsup_{r \to \infty} \frac{m(r, f)}{T(r, f)}.$$

It follows from the First Main theorem that $0 \leq \Delta(a, f) \leq 1$.
Ahlfors proved that the set of Valiron deficient values, those for which $\Delta(a, f) > 0$ has zero logarithmic capacity.
Hayman's result implies that every F_σ set of zero capacity can occur, even for an entire function.
Valiron's deficiencies have no direct relation to the Second Main theorem. To explore consequences of the Second Main theorem, Nevanlinna introduced another notion of deficiency:

$$\delta(a, f) = \liminf_{r \to \infty} \frac{m(r, a)}{T(r, f)}.$$

Then the Second Main theorem implies that the set of deficient values is at most countable, and the following *defect relation* holds:

$$\sum_{a \in \overline{\mathbf{C}}} \delta(a, f) \leq 2.$$

The question arises whether one can say anything else about deficiencies except this relation. This question is called the Inverse Problem of value distribution theory. Until 1953 it was even unknown whether the set of deficient values is finite, first such example was constructed by A. A. Goldberg.
In the paper of Fuchs and Hayman (1962) the Inverse problem was completely solved for entire functions: any deficiencies such that

$$\sum_{a \in \mathbf{C}} \delta(a, f) \leq 1$$

can occur for entire functions.
For meromorphic functions, this problem was solved by Drasin in 1976, who showed that in general, nothing can be said except the defect relation.

The most famous result of Hayman in the theory of meromorphic functions (and perhaps the most famous of all his results) is what is called Hayman's Alternative (Ann. Math., 1959).

A meromorphic function either takes every finite value infinitely often, or every derivative takes every finite non-zero value infinitely often.

In other words, *the two conditions* $f(z) \neq a$ *and* $f^{(n)}(z) \neq b$, *where* $a \in \mathbf{C},\ b \in \mathbf{C}^*$ *and* $n \geq 1$, *then* $f = \mathrm{const}$.
Notice that Picard's theorem has 3 conditions on omitted values which imply that a function is constant, while this has only two.
Hayman's alternative is derived from a quantitative result:

$$T(r,f) \leq \left(2 + \frac{1}{n}\right) N(r,0) + \left(2 + \frac{2}{n}\right) N(r,1,f^{(n)}) + S_n(r,f).$$

The proof of this consists of a formal algebraic manipulations with Nevanlinna characteristics, and it makes an impression of pure magic. It is exactly this feature which makes it possible to extend this proof to other situations, see below.
I was never able to understand how one could find such a proof.
Besides the main result this paper of Hayman contains many other results and conjectures which determined the development of the subject for many years. Two most important conjectures were

Conjecture 1. *If for a meromorphic function f, f and f'' have no zeros then f is an exponential or $f(z) = (az + b)^{-n}$.*

Conjecture 2. *For a transcendental meromorphic function f, ff' takes every finite value infinitely many times.*

Conjecture 1 was proved by Langley in 1993, and Conjecture 2 by Bergweiler and Eremenko in 1995. Both proofs required introduction of new methods, and many generalizations of these theorems were obtained.

Despite an enormous number of papers which deals with applications and generalizations of this theorem, the following basic question remains unsolved.
The general philosophy is that in theorems of Picard type, the condition that f omits a value can be replaced by the condition that all a-points have sufficiently high multiplicity. So what exactly are these multiplicities in Hayman's alternative?
Bergweiler and Langley (2005) proved that if all zeros of a meromorphic function have multiplicities $\geq m$ and 1-points of the derivative have multiplicities $\geq n$, then

$$\frac{7}{m} + \frac{8}{n} < 1$$

implies that the function is constant.

However this is not best possible since Nevo, Pang and Zalcman, proved in 2007 that

A meromorphic function whose all zeros are multiple and derivative omits 1 *must be constant.*

This strengthens both the Hayman alternative and the result of Bergweiler–Eremenko on Conjecture 2 mentioned above.
So the question about optimal multiplicities in Hayman's alternative remains open. Same applies to Langley's theorem:
Suppose that all zeros of a meromorphic function have multiplicities at least m and all zeros of f'' have multiplicities at least n. For which m and n one can conclude that $(f/f')'$ is constant?
(The differential operator at the end is the one that makes exceptional functions in Langley's theorem constant.)

Of all applications of Hayman's alternative, I will discuss only one, which is the most striking one, on my opinion. This application is to the conjecture of Wiman from the theory of entire functions, which does not look related to the value distribution of meromorphic functions.

Wiman's conjecture says that *for a real entire function* f*, the condition that all zeros of* ff'' *are real implies that* f *is in the Laguerre–Pólya class (the closure of polynomials with all zeros real).*

This is now proved by a combined effort of several authors spread over 1960-2003. The first breakthrough is due to Levin and Ostrovskii (This was the subject of Ostrovski's PhD thesis and Levin was his advisor).

The idea was that if ff'' has only real zeros, then $g := f/f'$ is meromorphic in the upper half-plane H and omits 0, and $g' = 1 - ff''/(f')^2$ omits 1. This is similar to the assumptions of the Hayman alternative, and one would like to derive some estimate of g.

The first attempt to obtain an analog of Nevanlinna theory for functions in a half-plane is due to Nevanlinna himself. He exhausted the half-plane by semi-disks, and his definition of characteristic consisted of three terms instead of two. Unfortunately these characteristics proposed by Nevanlinna do not share the main formal properties of the characteristics for the plane. (A. A. Goldberg showed in 1975 that the analog of the Lemma on the logarithmic derivative does not hold for Nevanlinna characteristics in a half-plane).

A more useful generalization of Nevanlinna theory for the half-plane is due to Levin-Ostrovskii and Tsuji. These are called Tsuji characteristics and they are based on a generalization of the Jensen formula for the exhaustion of the upper half-plane by the horocycles

$$\{z : |z - ir/2| \leq r/2.$$

Levin and Ostrovski proved that Tsuji characteristics possess all algebraic properties of the usual Nevanlinna characteristics, including the Lemma on the logarithmic derivative, and this permitted them to generalize Hayman's alternative to functions in the upper half-plane:

If g is meromorphic in the upper half-plane, has no zeros, and g' has no 1-points, then the Tsuji characteristic satisfies $\mathfrak{T}(r, f) = O(\log r)$.

This paper of Levin and Ostrovski was the foundation of all subsequent research on the Wiman conjecture (Hellerstein, Williamson, Shen, Sheil-small, Bergweiler, Eremenko, Langley) which culminated in the proof of this conjecture in 2005.
Let me also mention a related conjecture of Pólya on repeated differentiation. In a somewhat restated form it can be formulated as an alternative:

For an arbitrary real entire function f, either all zeros of all $f^{(n)}$ for $n > n_0$ are real, or the number of non-real zeros of $f^{(n)}$ tends to ∞ as $n \to \infty$.

The proof of this simple statement uses the Wiman conjecture, its generalization by Langley to higher derivatives, and a new version of the saddle point asymptotics inspired by another famous paper of Hayman, Generalization of Stirling's formula (1956).

Spherical derivative. Ahlfors' definition of characteristic involves the spherical derivative

$$f^{\#} = \frac{|f'|}{1 + |f|^2}.$$

The meaning of this expression is the infinitesimal length distortion when f is considered as a map from the plane with Euclidean metric to the Riemann sphere with the spherical metric. (Common normalization in Function theory assumes that the radius of the sphere is $1/2$, so the area is π.) Then

$$A(r, f) = \frac{1}{\pi} \int_{|z| \leq r} (f^{\#})^2 dxdy$$

is the average covering number of the sphere by the image of the disk $|z| \leq r$, and the Ahlfors characteristic is

$$T_0(r, f) = \int_0^r A(t, f) \frac{dt}{t}.$$

It follows immediately from the definition that

$$f^{\#}(z) = O(|z|^{\sigma}) \quad \text{implies} \quad T(r, f) = O(r^{2\sigma+2}),$$

and it is easy to see that $2\sigma + 2$ is the optimal exponent (consider a doubly periodic function, as a simple example).
It was an unexpected discovery of Hayman and Clunie, that for functions f omitting one value, this estimate can be improved: the same assumption implies $T(r, f) = O(r^{\sigma+1})$. This result confirms a principle conjectured in 1950 by Littlewood that the measure ${f^{\#}}^2 dxdy$ for an entire function cannot be uniformly spread in the plane: it is concentrated on a small subset. Look, for example at $f(z) = e^z$ where most of this measure lies in a horizontal strip.

The original proof of Clunie and Hayman was rather complicated, but now we have two very simple proofs: one is due to Pommerenke (1970), another to Barrett and Eremenko (2012). The authors of this last paper extend the result to holomorphic curves in the complex projective space. Let $f = (f_0, \ldots, f_n)$ be entire functions without zeros common to all of them. The Nevanlinna characteristic is defined analogously to the Ahlfors definition, where

$$f^{\#^2} = \|f\|^{-4} \sum_{0 \leq i < j \leq n} |f_i' f_j - f_i f_j'|^2,$$

this is the square of the length distortion by f from the Euclidean metric to the Fubini-Study metric in the complex projective space of dimension n. Then the theorem of Barrett and Eremenko says that

If f omits n hyperplanes in general position then $f^{\#}(z) = O(|z|^{\sigma})$ implies $T(r) = O(r^{\sigma+1})$.

When $n = 1$ we obtain the result of Clunie and Hayman.

A more striking generalization is due to da Costa and Duval. In dimension 1 it says

If f is a meromorphic function with $f^{\#} = O(r^{\sigma})$ then

$$T(r) = N(r, a) + O(r^{\sigma+1}).$$

To state their result in arbitrary dimension, we consider a holomorphic curve $f = (f_0, \ldots, f_n)$. Coordinates are assumed to be entire functions without zeros common to all of them. Then for every hyperplane a given by equation

$$a_0 w_0 + \ldots + a_n w_n = 0$$

the counting function $n(r, a)$ is the number of zeros of the entire function $(a, f) = a_0 f_0 + \ldots + a_n f_n$ in $\{z : |z| \leq r\}$, and $N(r, a)$ is defined in the usual way. Da Costa and Duval proved the following:

If a curve f satisfies $f^{\#} = O(r^{\sigma})$ *and omits* $n-1$ *hyperplanes then for any hyperplane a such that all n hyperplanes together are in general position,*

$$T(r,f) = N(r,a) + O(r^{\sigma+1}).$$

Without the assumption on omitted hyperplanes, they obtained under the same condition that $f^{\#} = O(r^{\sigma})$ that

$$(q-n+1)T(r,f) \le \sum_{j=1}^{q} N(r,a_j) + o(r^{2\sigma+2}),$$

which is meaningful only when $T(t,f) \neq o(r^{2\sigma+2})$. This should be compared with the weak Cartan's Second Main theorem which has $(q-n-1)T(r,f)$ in the left hand side, but a much better error term.
Whether one can prove the error term to $O(r^{\sigma+1})$ in this result of Da Costa and Duval, is an interesting open problem.

Meromorphic functions and holomorphic curves in projective spaces with uniform bounds on their spherical derivatives became an important topic when Zalcman (1975) and Brody (1978) rescaling lemmas were proved.
Another reason is that they make one of the important examples for the theory of mean dimension of Gromov and Lindenstraus.

Derivatives. The question of relation between the asymptotic behavior of a meromorphic function and its derivative, besides its independent interest, is fundamental for Nevanlinna theory, where the main technical tool is the Lemma on the Logarithmic derivative:

$$m(r, f'/f) = S(r, f).$$

The question of comparison of $T(r, f)$ and $T(r, f')$ occupied Hayman for many years, and the deepest available results on this topic are contained in his three papers (1964, 1965, 1989), the last one is joint with Joe Miles. The main results are the following.

1. If $T(r, f) = O(\log^2 r)$ *then*

$$\frac{1}{2} \leq \liminf_{r\to\infty} \frac{T(r, f')}{T(r, f)} \leq 2.$$

This is best possible, but the lower estimate is attained on $f(z) = z^2$, and it is conjectured that for transcendental f the RHS should be 1. (This is the case for transcendental entire f).

2. There exist entire functions with $T(r, f) = O(\phi(r)\log^2 r)$ *with any* $\phi \to \infty$, *and such that*

$$T(r, f')/T(r, f) \to 0$$

on the set of r *of infinite logarithmic measure.*

This disproves a conjecture of Nevanlinna who hoped that $T(r, f')/T(r, f)$ greater than a positive constant, away from an an exceptional set.

3. For all transcendental meromorphic functions and for all $K > 1$

$$\frac{T(r, f')}{T(r, f)} > \frac{1}{2eK},$$

outside of a set of upper logarithmic density $\delta(K) < 1$.

These remarkable results still do not exhaust the subject, for example, the following old conjecture of Fuchs is still open:

For entire functions of order $< 1/2$, $\delta(f'/f, 0) = 0$

This is closely related to the statement that

$$\limsup_{r\to\infty} \frac{T(r, f')}{T(r, f)} = 1$$

for all entire functions of order $< 1/2$.