Accessory parameter of the Heun equation

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Consider the Heun equation

$$y'' + \left(\sum_{j=0}^{2} \frac{1 - \alpha_j}{z - a_j} \right) y' + \frac{Az - \lambda}{(z - a_0)(z - a_1)(z - a_2)} y = 0,$$

where the parameters α_j, a_j, A satisfy $\alpha_j > 0$,

$$A = \alpha' \alpha'', \quad \sum_{j=0}^{2} \alpha_j + \alpha' + \alpha'' = 2,$$

where α' and α'' are real. λ is called the *accessory parameter*.

Problem. For given a_j, α_j, A, describe the set of values of λ for which the projective monodromy group of the equation is conjugate to a subgroup of $PSU(2)$.

Same question when the parameters a_j and λ are also real.

For which parameters a_j, α_j, A is this set non-empty? Is it always finite? How many elements can it contain?

Update (May 2019). I proved that this set is finite,

A. Eremenko, Metrics of constant positive curvature with four conic singularities on the sphere, Proc. AMS 148, 9 (2020) 3957–3965; arXiv:1905.02537.

However this proof is non-constructive: no explicit upper estimate in terms of α_j, A is known, except for very special cases.