Homework 7

1. Find and classify the isolated singularities of the following functions in the complex plane. In the case of poles find their multiplicities:

a) $z/(1-\cos z)$;
b) $\cot z - 1/z$;
c) $z/(e^z - 1)$;
d) $\exp \tan z$;
e) $\sin(\exp(1/z))$;
f) $z^{-2}\exp(1/(1+z))$;

2. Prove the l'Hôpital rule for holomorphic functions: *If f and g are analytic in a neighborhood of the point a, and both have a root of multiplicity m at a, then*

$$\lim_{z\to a}\frac{f(z)}{g(z)} = \lim_{z\to 0}\frac{f^{(m+1)}(z)}{g^{(m+1)}(z)}.$$

3. Expand the following functions into Laurent series in the specified regions:

a) $z^2/(z^2+1), \quad |z|>1$;
b) $(z-1)^{-2}(z+2)^{-1}, \quad 1<|z|<2$;
c) $z^3(z+1)^{-1}(z-2)^{-1} \quad 0<|z+1|<3$;

4. How many roots does the polynomial z^4+z^3-4z+1 have in the ring $1<|z|<2$?

5. Prove that the equation $z\sin z = 1$ has only real roots. Hint: compute the number of roots in appropriate discs $|z|<r$, and then the number of real roots in the same discs.

6. How many non-real roots the equations $\tan z = az$ have for various values of $a>0$?

7. Using the partial fraction decomposition, give an algebraic proof of the fact that the sum of the residues of a rational function is zero. ("Algebraic" means "not using the Residue Theorem, or Cauchy Theorem, or any integrals").