Solutions to HW 3

1 and 2b are just direct computations.
2a). For a monomial $M = ax^n y^k$ the degree is $m = k + n$. We have $M_x = nax^{n-1}y^k$ and $M_y = kax^n y^{k-1}$, so $xM_x + yM_y = (k+n)ax^n y^k = (k+n)M = mM$, as required. Arbitrary homogeneous polynomial P of degree m is just a sum of monomials, each of the same degree m. Adding the equations $xM_x + yM_y = mM$ for each monomial in P we obtain $xP_x + yP_y = mP$.
2c). $(x^3 - 2xy^2)^{-1}$.
3a). An integrating factor is $e^{2x} = \exp \int_0^x 2dx$. Thus we have

$$e^{2x}y' + 2e^{2x}y - e^{2x}x = 0, \quad \text{or} \quad (e^{2x}y)' = xe^{2x}.$$

By integrating by parts, we obtain

$$y = e^{-2x} \int_0^x te^{2t}dt = x/2 + e^{-2x}/4 - 1/4.$$

3b). An integrating factor is $e^{x^2} = \exp \int_0^x 2tdt$, so we have

$$e^{x^2}y' + 2xe^{x^2}y = (e^{x^2}y)' = x^3 e^{x^2}.$$

The RHS is integrated by a change of variable $w = x^2$, and the answer is

$$y = x^2/2 - 1/2 + e^{-x^2}/2.$$

4. Let $y(t)$ be the temperature of the turkey (the oven was turned off when $t = 0$). The temperature of the oven is $400 - 10t$. From the Law of Cooling we obtain

$$y' = 400 - 10t - y,$$

which is a linear equation. The integrating factor is e^t, and the general solution is

$$y(t) = 410 - 10t + Ce^{-t}.$$

From the initial conditions $C = -220$.
5. The area of a horizontal section is $S(h) = \pi r^2(h)$. Let the area of the hole be s, the speed of water near the hole $v(t)$, and the acceleration of gravity g. As we did in class, we obtain

$$S(h(t))h'(t) = sv(t) \quad \text{and} \quad gh = v^2/2,$$

the first equation is self evident, the second comes from the conservation of energy. Combining these equations we obtain

$$S(h)h' = s\sqrt{2gh},$$

which is separable. Actually we do not even need to integrate it, because we are only asked, when the solution $h(t)$ is linear, that is when $h' = \text{const}$. This happens if and only if $S(h)$ is proportional to $\sqrt{h}$, that is $r(h)$ is proportional to $h^{1/4}$. This is how a water clock usually looks.

6. From the picture we see

$$\tan^{-1}\frac{y}{x} = 2\tan^{-1}\frac{dy}{dx}. \tag{1}$$

Recalling from trigonometry that

$$\tan 2t = \frac{2\tan t}{1-\tan^2 t},$$

we apply tan to both sides of (1) and obtain

$$1 - \left(\frac{dy}{dx}\right)^2 = 2\frac{x}{y}\frac{dy}{dx},$$

which is a quadratic equation with respect to dy/dx. Solving it we obtain

$$\frac{dy}{dx} = \frac{-x \pm \sqrt{x^2+y^2}}{y},$$

or

$$ydy + xdx = \pm\sqrt{x^2+y^2}dx,$$

as advertised. Using the hint we put $w = x^2 + y^2$, then

$$dw = \pm 2\sqrt{w}dx,$$

so

$$\sqrt{w} = \pm x + C.$$

By squaring and returning to our original variables, we get

$$x^2 + y^2 = (\pm x + C)^2 \quad \text{or} \quad y^2 = \pm 2Cx + C^2,$$

which describes a parabola.

This property of parabolic mirrors is well known and widely used in flashlights, headlights, searchlights, beacon lights and so on. Our argument shows that ONLY parabolic mirrors can direct light from a point source into a parallel beam.