Hermite–Van Vleck polynomials

October 4, 2015

We are interested in the special Heun equation, "special" means that all exponent differences are odd multiples of 1/2. Heun means 4 regular singularities.

It can be written on the sphere as

$$y'' + \left(\sum_{j=0}^{2} \frac{1-\alpha_j}{z-a_j} \right) y' + \frac{Az - \lambda}{(z-a_0)(z-a_1)(z-a_2)} y = 0, \tag{1}$$

where

$$A = \alpha'\alpha'',$$
$$\alpha' = (2 + \alpha_3 - \alpha_0 - \alpha_2 - \alpha_2)/2,$$
$$\alpha'' = (2 - \alpha_3 - \alpha_0 - \alpha_1 - \alpha_2)/2.$$

or on the torus

$$w'' - \left(\sum_{j=0}^{3} n_j(n_j+1)\wp(\zeta - \omega_j/2) + \lambda \right) w = 0, \tag{2}$$

where $\wp$ is the Weierstrass function with periods ω_1, ω_3, and $\omega_0 = 0, \omega_2 = (\omega_1 + \omega_3)/2$. The relation between parameters is

$$\alpha_j = n_j + 1/2 \tag{3}$$

The metric on the sphere has conic angles $2\pi\alpha_j$ radians, and the metric on the torus $4\pi\alpha_j$ radians.

The developing map is the ratio of two linearly independent solutions. We denote is $F = w_1/w_0$ of $f = y_1/y_0$. Equation (2) is related to (1) by the change of the independent variable $z = \phi(\zeta)$, where ϕ is the elliptic function with fundamental periods ω_1, ω_3 and critical values a_j. One of the standard normalizations is $a_j = e_{j+1}$, where $e_1 + e_2 + e_3 = 0$ but we will choose another normalization

$$(a_0, a_1, a_2) = (0, 1, a),$$

so parameter a will play the role of the modulus of the torus or a quadrilateral.

It was found by Hermite for the case of Lame equation, and by Darboux in general, that under the stated condition on the exponent differences, the *product* of two appropriate linealy independent solutions of (1) is a polynomial. (And the product of some linearly independent solutions of (2) is an elliptic function). According to Hermite and Darboux, two linearly independent solutions of (1) can be written as

$$y_0(z) = \sqrt{P} \exp\left(C \int \frac{\prod_{j=0}^{2} (z - a_j)^{\alpha_j - 1}}{P(z)} \right), \tag{4}$$

$$y_1(z) = \sqrt{P} \exp\left(-C \int \frac{\prod_{j=0}^{2} (z - a_j)^{\alpha_j - 1}}{P(z)} \right), \tag{5}$$

where P is the Hermite–Van Vleck polynomial. Let us write $P(z) = \prod_{j=1}^{d} (z - c_j)$. As y_0, y_1 have no singularities away from the a_j, the condition must be satisfied that all residues of the expression under the integral sign must be ± 1, that is

$$C \frac{\prod_{j=0}^{2} (c_k - a_j)^{\alpha_j - 1}}{\prod_{j \neq k} (c_k - c_j)} = \pm 1.$$

Hermite [3] found a way to solve this equation. He proved that P is a polynomial solution of the 3-d order linear ODE

$$W''' + 3pW'' + \left(p' + 2p^2 + 4q \right) W' + \left(4pq + 2q' \right) W = 0, \tag{6}$$

where

$$p(z) = \sum_{j=0}^{2} \frac{1 - \alpha_j}{z - a_j}$$

and

$$q(z) = \frac{Az - \lambda}{(z - a_0)(z - a_1)(z - a_2)}$$

are the coefficients of the Heun equation. He proved that (6) has a one-dimensional space of solutions consiting of polynomials of degree

$$d=\sum_{j=0}^{3}\alpha_j-2,$$

spanned by the Hermite–Van Vleck polynomial. This gives a simple algorithm of finding P. The coefficients of P are rational functions of a, λ (we always consider the α_j as fixed, and do not mention the dependence of all quantities of them).

We propose to call this polynomial an *Hermite–Van Vleck polynomial.* It is a polynomial in z whose coefficients depend on the a_j, α_j and λ as parameters.

Computation of spherical rectangles consists of two steps. First we compute the Hermite polynomial. Then for every (a, λ) formulas (4), (5) define a circular quadrilateral with developing map $f = y_0/y_1$. The condition that this circular quadrilateral is geodesic is equivalent to the condition That some period of the elliptic integral in (4,5) has zero real part.

We wrote a Maple program computing this Hermite-Van Vleck polynomial, and found many interesting features that we cannot explain.

First, it turns out that the coefficients P are polynomials of a and λ (no denominators).

Second, the matrix of the linear equation from which these coefficients are determined is lower triangular (rectangular) with five non-zero diagonals. The main diagonal consists of constants. In those cases when all these constants are non-zero, it is clear why the coefficients are poilynomials. But sometimes (for some α_j and a) some entries on the main diagonal are zeros.

We also computed the discriminant of P. One interesting feature is that this discriminant is highly factorizable, and factors are usually multiple.

We would like to better understand the properties of these Hermite-Van Vleck polynomials. Maple file with detailed comments en it accompanies this message.

Equation (2) for the first time in Darboux [1] as a generalization of Lamé's equation, and Darboux notices that Hermite's argument applies to it. Then it was studied by Van Vleck [7] in great detail, with the main emphasis on the properties of the Polynomial P, and the geometric interpretation with circular polygons. In the papers of Klein and Van Vleck, many graphs of the real curves $P(z, \lambda) = 0$ are drawn, for fixed a, and they look remarkable.

In 1940 Ince found that Lame's potential is finite gap, and in the modern times Treibich and Verdier, Gesztezy and K. Takemura and Veselov wrote many papers on this, but without any special attention to what we call Hermite-Van Vleck polynomials.

References

[1] G. Darboux, Sur une équation linéaire, C.R. 94 (1882) 1645–1648.

[2] A. Eremenko and A. Gabrielov, On metrics of curvature 1 with four conic singularities on tori and on the sphere, arXiv:1508.06510.

[3] Ch. Hermite, Sur l'équation de Lamé, Extrait des feuilles autographiées du Cours d'Analyse de l'École Polytechnique, 1-re Division, 1872-1873, 32-e leçon. Oeuvres de Charles Hermite, tome III, Paris, Gauthier-Villars, 1912, p. 118–122.

[4] F. Klein, Ueber den Hermite'schen Fall der Lamé'schen Differentialgleichnug, Math. Ann., 40 (1892) 125-129.

[5] A. A. Markov, On the zeros of Hermite's entire function and Lamé's functions, Comm. Kharkov Math. Soc., ser. 2, vol. 5, (1896) 74–80.

[6] E. G. C. Poole, Introduction to the theory of linear differential equations, Oxford, Clarendon Press, 1936.

[7] K. Takemura, On the Heun equation, Phil. Trans. R. Soc., A 2008 366, 1179–1201.

[8] E. Van Vleck, Onb certain differential equations of the second order allied to Hermite's equation, Amer. J. Math., 21 (1899) 126–167.

[9] E. Whittaker and G. Watson, A course of modern analysis, vol. II. 4-th edition, Cambridge University Press, 1927.