Homework 3

1. Integrating factors for linear equations. It was shown in class that $I(x) = \exp \int p(x)dx$ is an integrating factor for the linear equation

$$dy + (p(x)y + q(x))dx = 0. \tag{1}$$

This means that the equation

$$Idy + I(py + q)dx = 0$$

is exact. Apply the usual method of solving exact equations to derive a formula for the general solution of (1).

2. Integrating factors for homogeneous equations. A function $f(x, y)$ is called "homogeneous of degree m" if for every $k \neq 0$ we have $f(kx, ky) \equiv k^m f(x, y)$. So a "homogeneous function" is a "homogeneous function of degree 0."

a) Show that a polynomial $P(x, y)$ is homogeneous of degree m if and only if it satisfies the identity

$$xP_x(x, y) + yP_y(x, y) \equiv mP(x, y).$$

(This fact is due to Euler, it is actually true for all differentiable homogeneous functions, not only for polynomials). Hint: consider a monomial first.

b) If P and Q are two homogeneous polynomials of the same degree, then the equation

$$Pdx + Qdy = 0 \tag{2}$$

is homogeneous. Show that an integrating factor for this equation is

$$I = \frac{1}{xP + yQ}.$$

c) Find an integrating factor for

$$(x^2 - y^2)dx - xydy = 0.$$

3. Solve the following initial-value problems:

a) $y' + 2y - x = 0, \quad y(0) = 0;$
b) $y' + 2xy = x^3, \quad y(0) = 0.$

4. A Turkey in the oven. A turkey was cooked in a 400° oven until the turkey reached 180°, at which point the oven was turned off. After that moment the oven's temperature decreases at a constant rate of 10° per minute. Find the temperature of the turkey as a function of time after the oven was switched off, assuming that the coefficient of proportionality in the Law of Cooling is equal to 1.

5. Design a water clock. Before the invention of mechanical clock (in XIII century, in Germany) the most precise available time measuring instrument was water clock (clepsydra). A typical water clock consists of a transparent vessel in the shape of a surface of revolution with vertical symmetry axis, and with a little hole in the bottom. This surface can be described by a function $r(h)$, the radius of horizontal cross-section on height h. Find what $r(h)$ should be, for the water level in the vessel to decrease at constant speed. Sketch the shape of the vessel you found. (If you have ever seen a water (or sand) clock, it probably had this shape).

6. Curved mirror for a searchlight. A mirror has the shape of a surface of revolution around the x-axis. The radius of vertical cross-sections is given by a function $y = f(x)$. The mirror reflects all rays coming from the origin so that the reflected rays are parallel to the x axis. Find the shape of this mirror, that is the function $f(x)$.

Hint: your calculations should lead you to a differential equation, equivalent to

$$xdx + ydy = \pm\sqrt{x^2 + y^2}dx.$$

One way to solve it is to notice that the LHS equals $d(x^2 + y^2)/2$, which suggests to introduce a new variable $w = x^2 + y^2$.