Homework 8

1. Expand the following functions into partial fractions:

$$\frac{z^4}{z^3-1}, \quad \frac{1}{z(z+1)^2(z+2)^3}.$$

2. If Q is a polynomial with distinct roots $\alpha_1, \ldots, \alpha_n$, and if P is a polynomial of degree less than n, prove that

$$\frac{P(z)}{Q(z)} = \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)(z-\alpha_k)}.$$

3. If a meromorphic function is m-to-one in a neighborhood of some point a, and $m > 1$, we say that a is a *critical point of index* $m-1$. Prove that the sum of the indices of all critical points (on the Riemann sphere) of a rational function of degree d is always $2d-2$.

4. Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^4}.$$

5. Find the sum of the series

$$\sum_{n=-\infty}^{\infty} \frac{1}{z^3-n^3}.$$

6. Find the partial fraction decomposition of $1/\cos \pi z$ and use it to prove the formula

$$\pi/4 = 1 - 1/3 + 1/5 - 1/7 \ldots.$$

7. Prove that for $|z| < 1$

$$(1+z)(1+z^2)(1+z^4)(1+z^8)\ldots(1+z^{2^n})\ldots = \frac{1}{1-z}.$$

You have to give a complete justification, why is the product convergent etc.