Additional problems, HW 9

1. If (z_0, z_1, z_2) and (w_0, w_1, w_2) are two triples of distinct complex numbers then the fractional-linear transformation sending z_j to w_j can be found from the formula

$$\frac{z - z_0}{z - z_1}\frac{z_3 - z_1}{z_3 - z_0} = \frac{w - w_0}{w - w_1}\frac{w_3 - w_1}{w_3 - w_0}$$

which was proved in class. Prove that if one of the z_j and/or one of the w_j is ∞, the following rule should apply: "cross out all factors that contain infinity".

2. Suppose that two disjoint circles on the Riemann sphere are given. Prove that there exists a fractional-linear transformation which sends them to two concentric circles. (Concentric circles are understood as in elementaty geometry: ordinary circles with the same center).

3. Prove that every rational function which maps the unit circle into itself is of the form

$$\lambda \prod_{k=1}^{d} \frac{z - a_k}{1 - \overline{a_k} z},$$

where $|\lambda| = 1$ and $|a_k| \neq 1$ for all k. When does this function also maps the unit disc into itself?