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Geometric theory of meromorphic functions

A. Eremenko*

Abstract

This is a survey of results on the following problem. Let X be a simply connected Riemann surface spread over the Riemann sphere. How are the properties of the uniformizing function of this surface related to the geometric properties of the surface?

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1. According to the Uniformization Theorem, for every simply connected Riemann surface X there exists a conformal homeomorphism $\phi : X_0 \to X$, where X_0 is one of the three standard regions, the Riemann sphere $\overline{\mathbf{C}}$, the complex plane $\mathbf{C}$ or the unit disc $\mathbf{U}$. We say that the *conformal type* of X is *elliptic, parabolic* or *hyperbolic*, respectively. The map ϕ is called the *uniformizing map.* If X is given by some geometric construction, the problem arises to relate properties of ϕ to those of X. This includes the determination of the conformal type of X [11, 31, 40, 41]. The case which was studied most is when $X \subset \mathbf{C}$ is a simply connected region, $X \neq \mathbf{C}$. Then X is of hyperbolic type and ϕ is a univalent function in $\mathbf{U}$.

We recall a more general construction. A surface *spread over the sphere* is a pair (X, p), where X is a topological surface and $p : X \to \overline{\mathbf{C}}$ a continuous, open and discrete map. This map p is usually called the *projection.* The natural equivalence relation is $(X, p) \sim (Y, q)$ if there is a homeomorphism $\phi : X \to Y$ with the property $p = q \circ \phi$. According to a theorem of Stoïlov, every continuous open and discrete map p between surfaces locally looks like $z \mapsto z^n$. Those points where $n > 1$ are isolated, they are called *critical points.* Stoïlov's theorem implies that there is a unique conformal structure on X which makes p holomorphic. If ϕ is the uniformizing map, then $f = p \circ \phi$ is a meromorphic function in one of the three standard regions $\overline{\mathbf{C}}, \mathbf{C}$ or $\mathbf{U}$. The surface (X, p) spread over the sphere is then the "Riemann surface of f^{-1}".

If D is a region on the sphere, a *branch* of p^{-1} in D is a continuous function $\psi : D \to X$ such that $p \circ \psi = \mathrm{id}_D$.

*Department of Mathematics, Purdue University, West Lafayette IN 47907, USA, E-mail: eremenko@math.purdue.edu. Supported by NSF grant DMS 0100512 and by the Humboldt Foundation.

We can define the length of a curve in X as the spherical length[1] of its image under p. Then X becomes a metric space with an *intrinsic metric*, which means that the distance between two points is the infimum of the lengths of curves connecting these points. Similarly, if $p: X \to \mathbf{C}$, and the Euclidean metric in $\mathbf{C}$ is used to measure lengths of curves, we obtain a Riemann surface *spread over the plane.*

The intrinsic metric on X is a smooth Riemannian metric of constant curvature on the complement of the critical set of p. It is easy to show that the intrinsic metric on X determines the projection p up to an isometry of the image sphere or the plane. In what follows, unless otherwise stated, (X, p) denotes a simply connected surface spread over the sphere, equipped with the intrinsic spherical metric.

Some criteria of conformal type can be stated in terms of topological (or even set-theoretic) properties of p. For example, Picard's theorem implies that X is of hyperbolic type if p omits three points.

1.1 Ahlfors's Five Islands Theorem. *Suppose that for five Jordan regions with disjoint closures on the sphere, there are no branches of p^{-1} in any of these regions. Then X is of hyperbolic type.*

This theorem was stated for the first time by Bloch [6] (with discs instead of Jordan regions) and proved by Ahlfors in [2]. A short and simple proof was recently found by Bergweiler [3].

Sullivan asked the general question, for which surfaces (X, p) the conformal type is determined by topological properties of p. More precisely, let us say that a simply connected surface (X, p) spread over the sphere has a *stable type* if for every homeomorphism $\psi : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ the surface $(X, \psi \circ p)$ has the same type as (X, p). For example, surfaces satisfying the conditions of the Five Islands Theorem are of stable hyperbolic type. Another interesting class of surfaces of stable type is the *Speiser class S*. We say that $(X, p) \in S$ if there exists a finite set $A \subset \overline{\mathbf{C}}$ such that the restriction $p : X\backslash\pi^{-1}(A) \to \overline{\mathbf{C}}\backslash A$ is a covering. The stability of type of such surfaces was proved by Teichmüller in [37] as one of the first applications of quasiconformal mappings. His argument extends to the somewhat wider class consisting of surfaces with the property that the distances between their singularities[2] are bounded from below by a positive constant. All surfaces of stable parabolic type known to the author have this property. The simplest example of a parabolic surface with a non-isolated singularity is $(\mathbf{C}, p)$, where $p(z) = \sin z/z$, and it follows from a result of Volkovyskii [40, Th. 45] that the conformal type of this surface is not stable.

2. If we take the five regions to be spherical discs of equal radii, Theorem 1.1 implies the following [1]: *Suppose that for some $\epsilon > 0$ there are no branches of p^{-1} in discs of radii $\pi/4 - \epsilon$. Then X is of hyperbolic type.* The question arises, what is the best constant for which this result still holds. Let $B(X, p)$ be the supremum of radii of discs where branches of p^{-1} exist, and $B = \inf B(X, p)$, where the infimum is taken over all surfaces of elliptic or parabolic type. Ahlfors's estimate $B \geq \pi/4$ was improved by Pommerenke [33] to $B \geq \pi/3$, and recently the sharp result was

[1] We choose the spherical length element to be $2|dz|/(1 + |z|^2)$, so that the curvature of the spherical metric is $+1$.

[2] A formal definition of singularities and distance between them is in section 2.

obtained in [8]:

$$B = b_0 := \arccos(1/3) \approx 0.39\pi.$$

We have $B(\mathbf{C}, \wp) = B$, where $\wp$ is the Weierstrass function of a hexagonal lattice. It is interesting to notice that $B = b_0$ implies Theorem 1.1 by a simple argument given in [8].

For surfaces (X, p) of elliptic type we have $B(X, p) > b_0$, but it is not known whether the constant b_0 is best possible in this inequality.

This result for elliptic surfaces is derived from the Gauss–Bonnet theorem. The proof of $B \geq b_0$ for parabolic surfaces is more complicated. For our class of surfaces with intrinsic metric, one can define integral curvature [35] as a signed Borel measure on X which is equal to the area on the smooth part of X and has negative atoms at the critical points of p. The assumption that $B(X, p) < b_0$ implies that the atoms of negative curvature are sufficiently dense on the surface, so that on large pieces of X the negative part of the curvature dominates the positive part. Then a bi-Lipschitz modification of the surface is made, which spreads the integral curvature more evenly on the surface, resulting in a surface whose Gaussian curvature is bounded from above by a negative constant, and the Ahlfors–Schwarz lemma implies hyperbolicity. A non-technical exposition of the ideas of this proof is given in the survey [7] which contains some further geometric applications of this technique of spreading the curvature by bi-Lipschitz modifications of a surface.

3. To formalize the notion of a singular point of a multi-valued analytic function, Mazurkiewicz [30] introduced another metric, $\rho(x, y) = \inf\{\operatorname{diam} p(C)\}$ on X, where diam is the diameter with respect to the spherical metric and the infimum is taken over all curves $C \subset X$ connecting x and y. Every point $x \in X$ has a neighborhood where the Mazurkiewicz metric coincides with the intrinsic one, but in general the Mazurkiewicz metric is smaller. Let X^* be the completion of X with respect to the Mazurkiewicz metric. Then p has a unique continuous extension to X^*. The elements of the set $Z = X^* \backslash X$ are called *transcendental singularities* of (X, p). The *algebraic singularities* are just the critical points of p.

To each transcendental singularity corresponds an *asymptotic curve* $\gamma : [0, 1) \to X$ which has no limit in X but its image $p \circ \gamma$ has a limit in $\overline{\mathbf{C}}$. This limit is called an *asymptotic value* and it is the projection of the singularity.

If X is of parabolic type, then the set of singularities is totally disconnected. This can be proved by using Iversen's theorem [24, 31]. The following classical result [31], which implies Iversen's theorem, shows that if we "look in all directions from a point" on a parabolic surface then very few singularities are visible.

3.1 GROSS'S THEOREM. *Let (X, p) be a simply connected surface of parabolic type spread over the sphere, and $x \in X$. Then X contains a geodesic ray from x of length π (that is from x to the "antipodal point") in almost every direction.*

It is not known whether the estimate of the size of the exceptional set in this theorem can be improved, but there are examples where this exceptional set of directions has the power of the continuum [40, Th. 17].

The projection of the set Z of singular points is an analytic (Suslin) set [30], and for every analytic subset A of the sphere one can find surfaces of both parabolic and hyperbolic types for which $A = p(Z)$ [21].

The following classification of transcendental singularities was introduced by Iversen [24]. A singular point $x \in X^*\backslash X$ is called *direct* if for some neighborhood $V \subset X^*$ of x, the map p omits $\pi(x)$ in $V\backslash\{x\}$. Otherwise x is called *indirect.* For example, $(\mathbf{C}, \sin z/z)$ has two direct singularities over ∞ and two indirect singularities over 0. The following result was proved in [22]:

3.2 Heins's Theorem. *For a parabolic Riemann surface spread over the sphere, the set of projections of direct singularities is at most countable.*

On the other hand, for some parabolic surfaces, the set of direct singularities lying over one point may have the power of the continuum.

To state further results on direct singularities we recall the notion of the order of a meromorphic function in the plane. Let (X, p) be an open simply connected Riemann surface of parabolic type, spread over the sphere. Then the intrinsic metric defines a notion of area on X. If ϕ is the uniformizing map and $f = p \circ \phi$ the corresponding meromorphic function in $\mathbf{C}$, then the "average covering number" of the sphere by the images of the discs $D(r) = \{z : |z| \leq r\}$ is defined as the area of $\phi(D(r))$ divided by the area of the sphere $\overline{\mathbf{C}}$, which is the same as

$$A(r, f) = \frac{1}{\pi} \int_{D(r)} \frac{|f'|^2}{(1 + |f|^2)^2}, \quad r \geq 0,$$

and the order of f is defined as

$$\lambda(f) = \limsup_{r \to \infty} \frac{\log A(r, f)}{\log r}.$$

It is easy to verify that the order depends only on (X, p) rather then on the choice of the uniformizing map ϕ.

3.3 Denjoy–Carleman–Ahlfors Theorem. *If f is a meromorphic function in the plane, and the Riemann surface $(\mathbf{C}, f)$ has $k \geq 2$ direct singularities, then*

$$\liminf_{r \to \infty} r^{-k/2} A(r, f) > 0.$$

If p omits a point $a \in \overline{\mathbf{C}}$, then a theorem of Lindelöf implies that at least half of all singularities of (X, p) lie over a, and these singularities are evidently direct. So we obtain that such surface can have at most $2\lambda(f)$ singularities over the points in $\overline{\mathbf{C}}\backslash\{a\}$. As a corollary, an entire function of order λ can have at most 2λ finite asymptotic values. This is in contrast with the case of meromorphic functions: there exist meromorphic functions in the plane of arbitrary prescribed order $\lambda \geq 0$ for which every point on the sphere is an asymptotic value [13].

In the simplest example $\sin z/z$ mentioned above, the indirect singularities over 0 are accumulation points of critical points. A result of Volkovyski [40, Th. 17] shows that this is not always so: there are parabolic surfaces without critical points, having indirect singularities. However, for functions of finite order, the following theorem was proved in [4]:

3.4 Theorem. *Let f be a meromorphic function of finite order, and let a be an indirect singularity of $(\mathbf{C}, f)$. Then every neighborhood of a contains critical points z such that $f(z) \neq f(a)$.*

This theorem can be used to prove the existence of critical points under certain circumstances, and more generally, to study the value distribution of derivatives. In [4] it was used to prove a result which was conjectured by Hayman: for every transcendental meromorphic function f in $\mathbf{C}$, the equation $ff' = c$ has infinitely many solutions for every $c \in \mathbf{C}\backslash\{0\}$. There is no growth restriction on f in this last result.

Goldberg and Heins independently noticed that in Theorem 3.3 one may count some indirect singularities, so-called K singularities, together with the direct ones. However a geometric characterization of K-singularities is known only for a very special class of symmetric surfaces [17].

4. There are few instances when a precise correspondence can be established between classes of surfaces spread over the sphere and classes of meromorphic functions corresponding to them. We mention first a class of hyperbolic surfaces, spread over the plane.

4.1 Theorem. *For a surface (X, p) spread over the plane, the following conditions are equivalent:*
(a) The (Euclidean) radii of discs where branches of p^{-1} exist are bounded.
(b) A linear isoperimetric inequality holds on X.
(c) X is of hyperbolic type and the uniformizing map $\phi : \mathbf{U} \to X$ is uniformly continuous with respect to the hyperbolic metric on $\mathbf{U}$ and the intrinsic (Euclidean) metric on X.

The equivalence between (a) and (c) is essentially Bloch's theorem [5]. For the equivalence of (b) to the other two conditions the reference is [9] where Theorem 4.1 is stated for a more general classes of surfaces with intrinsic metric (not necessarily spread over the plane). Holomorphic functions $f = p \circ \phi$, where ϕ satisfies (c), are called Bloch functions. This class is important because of its connection with univalent functions: if g is univalent in $\mathbf{U}$, then $\log g'$ is a Bloch function, and every Bloch function has the form $c \log g'$ where g is univalent in $\mathbf{U}$ and c is a constant [33]. In the case of meromorphic functions and surfaces spread over the sphere, conditions (b) and (c) are still equivalent; the implication (b)$\to$(c) is due to Ahlfors, and the inverse implication to B. Kleiner (unpublished). It is not known what conditions could replace (a) in the spherical case. Functions satisfying (c) with the spherical metric instead of the Euclidean one are called *normal*, and they were much studied, see, for example [34].

Passing to parabolic surfaces, we first mention the following result of R. Nevanlinna [32].

4.2 Theorem. *For a surface (X, p) spread over the sphere, the following conditions are equivalent:*

(a) The set of transcendental singularities of (X, p) is finite, and p has no critical points.

(b) (X,p) is equivalent to $(\mathbf{C},f)$ where f is a solution of the differential equation

$$f'''/f' - (3/2)\,(f''/f')^2 = P,$$

where P is a polynomial.

This was extended by Elfving [12] to the case when p has finitely many algebraic and transcendental singularities. Every solution f of the Schwarz differential equation in (b) is a ratio of two linearly independent solutions of the linear differential equation $w'' + (P/2)w = 0$. This provides very precise information on the asymptotic behavior of f.

In the case of infinitely many algebraic singularities, one cannot obtain such complete results, but still for many subclasses of surfaces of the Speiser class (which was defined in Section 1) one can obtain rather complete information about the asymptotic behavior of the uniformizing functions [10, 16, 18, 27, 41]. We mention here a question asked by A. Epstein. Let $(\mathbf{C},f)$ be a parabolic surface of Speiser class, such that f is of finite order. If ψ is a homeomorphism of the Riemann sphere, then $(\mathbf{C},\psi\circ f)$ is parabolic because as we mentioned before, Speiser surfaces have stable type. Let ϕ be the uniformizing map of this deformed surface, so that $g = \psi\circ f\circ\phi$ is a meromorphic function in the plane. It is easy to show that g also has finite order. The question is whether the orders of f and g are the same. Künzi [27] showed that this is not necessarily so for meromorphic functions f, but the question remains unsolved for entire functions.

Our third example consists of a class of surfaces spread over the plane and having a symmetry property. Suppose that an anticonformal involution $s : X \to X$ is given. The set of fixed points of s will be called the axis. One can always choose the uniformizing map ϕ so that it conjugates the involution with the reflection with respect to the real axis. We say that (X,p) is *symmetric* if $p\circ s = \overline{p}$ where the bar stands for the complex conjugation. A symmetric surface spread over the plane is called a *MacLane surface* if all its singularities, algebraic or transcendental, belong to the closure of the axis. Evidently, there can be at most two such transcendental singularities.

The corresponding class of functions is related to entire functions of Laguerre–Pólya (LP) class: these are the real entire functions which are limits of real polynomials with real zeros. According to Laguerre and Pólya, this class LP has the parametric representation:

$$\exp(-az^2 + bz + c)\prod_k \left(1 - \frac{z}{a_k}\right) e^{z/a_k}, \quad a \geq 0,\ b, c, a_k \in \mathbf{R}.$$

4.3 MacLane's Theorem. *Every surface of MacLane's class is parabolic. The derivatives of the corresponding entire functions constitute the class LP.*

This was proved in [28]. A very illuminating geometric proof is given in [39]. The class LP and its geometric characterization occur in many questions of analysis.

The most striking application is to the spectral theory of second order differential operators on the real line with periodic potentials (Hill operators). It was discovered by Krein [26] that Lyapunov functions of periodic strings are exactly those real entire functions f of genus zero, with positive roots, which have the property that all solutions z of the equation $f^2(z) = 1$ are non-negative. These functions constitute a subclass of LP which can explicitly be described in terms of the corresponding MacLane surfaces.

Developing Krein's idea, Marchenko and Ostrovskii [29] obtained a parametrization of self-adjoint periodic Hill operators in terms of their spectral data. In a recent paper [38], Tkachenko extended this result to some non self-adjoint Hill operators (with complex potentials) by considering small perturbations of MacLane surfaces which are no longer symmetric, and etablishing an exact correspondence between a class of entire functions and a class of surfaces in the spirit of Theorem 4.3.

We finish this section by mentioning an unsolved problem about LP entire functions which is sometimes called a Pólya–Wiman conjecture, though apparently it was stated for the first time in [25].

It is evident from the definition that all derivatives of a function of the class LP have only real zeros. The converse is also true: if f is a real entire function, and all zeros of $ff'f''$ are real, then $f \in$ LP [23]. Levin and Ostrovskii [25] conjectured that it is enough to require that only f and f'' have real zeros. They proved that real entire functions with this property satisfy $\log\log|f(z)| \leq O(|z|\log|z|)$. On the other hand, Sheil-Small [36] proved the conjecture of Levin and Ostrovskii under the growth restriction $\log\log|f(z)| \leq O(\log|z|)$, so a gap remains.

5. No satisfactory analog of Theorem 4.3 is known for meromorphic functions and open surfaces spread over the sphere. However, there is a related result for *rational functions* which was used in [14] to prove a special case of an intriguing conjecture in real algebraic geometry.

5.1 Theorem. *If all critical points of a rational function f belong to a circle C on the Riemann sphere, then $f(C)$ is a subset of a circle.*

Let us call two rational functions f and g equivalent if $f = \ell \circ g$ where ℓ is a fractional linear transformation. Equivalent functions have the same critical points. We may assume without loss of generality that the circle C in Theorem 5.1 is the real line. Then Theorem 5.1 says that whenever the critical points of a rational function are real, it is equivalent to a real rational function. A rational function with prescribed critical points can be obtained as a solution of a system of algebraic equations, so Theorem 5.1 implies that all solutions of this system are real whenever the coefficients are real. We will see in a moment the geometric significance of this system of algebraic equations.

It is very easy to prove Theorem 5.1 for polynomials, because every two polynomials with the same critical points are equivalent (which means that solution of the system of algebraic equations mentioned above is essentially unique in this case). This is not so for rational functions: it turns out that for given $2d-2$ points in general position on the sphere there are finitely many, namely

$$u_d = \frac{1}{d}\binom{2d-2}{d-1}, \quad \text{the } d\text{-th Catalan number,}$$

of classes of rational functions of degree d which share these critical points. This is due to L. Goldberg [19] who reduced the problem to the following classical problem of enumerative geometry: given $2d-2$ lines in general position in (complex) projective space, how many subspaces of codimension 2 intersect all these lines? The answer to this last problem, the Catalan number u_d, was obtained by Schubert in 1886 who invented what is now known as "Schubert Calculus" to solve this and similar enumerative problems.

Notice that rational functions exhibit a very special property in Goldberg's result: as we mentioned above, there is only one class of polynomials with prescribed critical points, and similarly there is only one class of Blaschke products with prescribed critical points [42].

The general question of how many solutions to a problem of enumerative geometry can be real was asked by Fulton in [15]. A specific conjecture about the problem of finding subspaces of appropriate codimension intersecting given real subspaces was made by B. and M. Shapiro. For the problem of Schubert calculus stated above, this conjecture says that if all $2d-2$ given lines are tangent to the rational normal curve $z \mapsto (1 : z : \ldots : z^d)$ at real points, then all u_d subspaces of codimension 2 which intersect these lines are real (can be defined by real equations). This statement is equivalent to Theorem 5.1.

The proof of Theorem 5.1 is based on an explicit description of surfaces spread over the sphere which correspond to real rational functions with real critical points in the spirit of Vinberg's work about the MacLane's class mentioned in section 4.

Let R be the class of *real* rational functions f whose all critical points are real and simple. To describe the Riemann surface of f^{-1} we consider the *net* γ_f which is the preimage $\gamma_f = f^{-1}(\mathbf{R} \cup \infty)$ modulo homeomorphisms of the Riemann sphere commuting with complex conjugation. A net consists of simple arcs which meet only at the critical points of f. To each of these arcs we prescribe a *label* equal to the length of its image under f. One can describe explicitly all labeled nets which may occur from this construction. It turns out that labeled nets give a parametrization of the class R of rational functions. This parametrization has an advantage that it clearly separates the discrete, topological parameter (the net) from the continuous parameters (the labeling), and it turns out that the set of possible labelings of a given net has simple topological structure: it is a convex polytope.

Using topological methods, we show in [14] that for every net and for every set of $2d-2$ points on the real line there exists a rational function of the class R with this net and these critical points. On the other hand, a simple combinatorial argument shows that the number of possible nets on $2d-2$ vertices is equal to the Catalan number u_d. Thus one obtains u_d classes of real rational functions of degree d with prescribed real critical points. Comparison with L. Goldberg's result shows that in fact we constructed *all* classes of rational functions with prescribed real critical points. Thus all such classes contain real functions.

References

[1] L. Ahlfors, Sur les fonctions inverses des fonctions méromorphes, *Comptes Rendus*, 194 (1932) 1145–1147.

[2] L. Ahlfors, Zur Theorie der Überlagerungsflächen, *Acta Math.*, 65 (1935) 157–194.

[3] W. Bergweiler, A new proof of the Ahlfors five islands theorem, *J. Analyse Math.*, 76 (1998) 337–347.

[4] W. Bergweiler and A. Eremenko, On the singularities of the inverse to a meromorphic function of finite order, *Revista mat. Iberoamer.*, 11 (1995) 355–373.

[5] A. Bloch, Les théorèmes de M. Valiron sur les fonctions entières, et la théorie de l'uniformisation, *Comptes Rendus*, 178 (1924) 2051–2052.

[6] A. Bloch, La conception actuelle de la théorie des fonctions entières et méromorphes, *Enseign. Math.*, 25 (1926) 83–103.

[7] M. Bonk, Singular surfaces and meromorphic functions, *Notices AMS*, to appear.

[8] M. Bonk and A. Eremenko, Covering properties of meromorphic functions, negative curvature and spherical geometry, *Ann. Math.*, 152 (2000) 551–592.

[9] M. Bonk and A. Eremenko, Uniformly hyperbolic surfaces, *Indiana Univ. Math. J.*, 49 (2000) 61–80.

[10] D. Drasin and A. Weitsman, Meromorphic functions with large sums of deficiencies, *Adv. Math.*, 15 (1975) 93–126.

[11] P. Doyle, Random walk on the Speiser graph of a Riemann surface, *Bull. AMS*, 11 (1984) 371–377.

[12] G. Elfving, Über eine Klasse von Riemannschen Flächen und ihre Uniformisierung, *Acta Soc. Sci. Fenn.*, 2 (1934) 1–60.

[13] A. Eremenko, On the set of asymptotic values of a meromorphic function of finite order, Engl. transl.: *Mat. Notes*, 24 (1979), 914–916.

[14] A. Eremenko and A. Gabrielov, Rational functions with real critical points and the B. and M. Shapiro conjecture in real enumerative geometry, *Ann. Math.*, 155 (2002).

[15] W. Fulton, *Introduction to intersection theory in algebraic geometry*, AMS, Providence, RI, second edition, 1996.

[16] A.A. Gol'dberg, On a class of Riemann surfaces, *Mat. Sbornik*, 49 (1959) 447–458.

[17] A.A. Gol'dberg, Some asymptotic properties of meromorphic functions, *Complex Variables*, 37 (1998) 225–241. (Translated from the Russian original of 1956).

[18] A.A. Gol'dberg and I.V. Ostrovskii, *Distribution of values of meromorphic functions*, Moskva, Nauka, 1970 (Russian).

[19] L. Goldberg, Catalan numbers and branched coverings of the sphere, *Adv. Math.*, 85 (1991) 129–144.

[20] W. Gross, Über die Singularitäten analytischer Funktionen, *Monatshefte Math. Phys.*, 29 (1918).

[21] M. Heins, The set of asymptotic values of an entire function, *12-e Skand. Matematikerkongr., Lund 1953*, Lund, 1954, 56–60.

[22] M. Heins, Asymptotic spots of entire and meromorphic functions, *Ann. Math.*, 66 (1957) 430–439.

[23] S. Hellerstein and J. Williamson, Derivatives of entire functions and a question

of Pólya, I, *Trans. AMS*, 227 (1977) 227–249; II, *ibid.*, 234 (1977) 497–503.

[24] F. Iversen, *Recherches sur les fonctions inverses des fonctions méromorphes*, Thèse, Helsingfors, 1914.

[25] B.Ya. Levin and I.V. Ostrovskii, The dependence of the growth of an entire function on the distribution of zeros of its derivative, *Sibirsk. Mat. Z.*, 1 (1960) 427–455 (Russian).

[26] M.G. Krein, On the inverse problems of the theory of filters and λ-zones of stability, *Doklady AN SSSR*, 93 (1953) 767–770 (Russian).

[27] H. Künzi, Zwei Beispiele zur Wertverteilungslehre, *Math. Z.*, 62 (1955) 94–98.

[28] G. MacLane, Concerning the uniformization of certain Riemann surfaces allied to the inverse-cosine and inverse-gamma surfaces, *Trans. AMS*, 62 (1947) 99–113.

[29] V.A. Marchenko and I.V. Ostrovskii, A characterization of the spectrum of the Hill operator, *Mat. Sbornik* 97 (1975) 540–606, 633–634.

[30] S. Mazurkiewicz, Sur les points singuliers d'une fonction analytique, *Fund. Math.*, (1931) 26–29.

[31] R. Nevanlinna, *Eindeutige analytische Funktionen*, Springer-Verlag, Berlin, 1953.

[32] R. Nevanlinna, Über Riemannsche Flächen mit endlich vielen Windungspunkten, *Acta Math.*, 58 (1932) 295–373.

[33] Ch. Pommerenke, *Boundary behavior of conformal maps*, Springer, Berlin, 1992.

[34] Ch. Pommerenke, Estimates for normal meromorphic functions, *Ann. Acad. Sci. Fenn., Ser AI* 476 (1970).

[35] Yu.G. Reshetniak, *Two-dimensional manifolds of bounded curvature*, Geometry IV, Encyclopaedia of Math. Sci., 70, Springer, Berlin, 1993.

[36] T. Sheil-Small, On the zeros of the derivatives of real entire functions and Wiman's conjecture, *Ann. Math.*, 129 (1989) 170–193.

[37] O. Teichmüller, Eine Anwendung quasikonformer Abbildungen auf das Typenproblem, Deutsche Math., 2 (1937) 321–327.

[38] V. Tkachenko, Spectra of non-selfadjoint Hill's operators and a class of Riemann surfaces, *Ann. Math.*, 143 (1996) 181–231.

[39] E.B. Vinberg, Real entire functions with prescribed critical values, *Problems of group theory and homological algebra*, Yarosl. Gos. Un-t., Yaroslavl, 1989, 127–138 (Russian).

[40] L.I. Volkovyski, *Researches on the type problem of a simply connected Riemann surface*, Trudy Mat. Inst. im. V.A. Steklova, 34, 1950 (Russian).

[41] H. Wittich, *Neuere Untersuchungen über eindeutige analytische Funktionen*, Springer-Verlag, Berlin, 1955.

[42] S. Zakeri, On critical points of proper holomorphic maps of the unit disc, *Bull. LMS*, 30 (1998) 62–66.