Indicators of linear combinations

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Let f be an entire function of order ρ normal type. The indicator h_f is defined as

$$h_f(\theta) = \limsup_{r\to\infty} r^{-\rho} \log |f(re^{i\theta})|.$$

When this limit exists as $r \notin E$, where $E \subset [0, +\infty)$ is a set of zero density (which may depend on θ), we say that f has completely regular growth. Alternatively, functions of completely regular growth can be characterized by the property that the following limit exists in the space $D'(\mathbf{C})$ of Schwartz distributions:

$$\lim_{t\to\infty} t^{-\rho} \log |f(tre^{i\theta})| = r^\rho h_f(\theta).$$

It is known that h_f is a continuous 2π-periodic function which satisfies

$$h'' + \rho^2 h \geq 0$$

in the sense of Schwartz distributions $D'(\mathbf{R})$. Such functions are called ρ-trigonometrically convex. It is known [1] that for every ρ-trigonometrically convex function h, there exists an entire function f of order ρ and normal type whose indicator equals h.

Problem 1. *Let $A = \{a\}$, $a = (a_1, a_2, \ldots, a_n)$ be a set of vectors in $\mathbf{C}^n$ such that every n of them are linearly independent, and $a \mapsto h_a$ an assignment of some ρ-trigonometrically convex functions to every $a \in A$. Under what conditions one can find n entire functions of order ρ, normal type, such that the linear combinations*

$$g_a = a_1 f_1 + \ldots + a_n f_n$$

have indicators h_a?

An interesting version of this problem is also when there exist functions of completely regular growth with these properties.

A necessary condition is

$$\max_I h_a = \max_A h_a \qquad \text{for every} \quad I, \quad \text{such that} \qquad |I| = n. \tag{1}$$

For $n = 2$, this condition seems to be also sufficient, but for $n \geq 3$ this is not so.

When $\rho = 1$, the problem is simplified by using Laplace transform. For a function f of order 1, normal type, let

$$F(z) = \int_0^\infty e^{-\zeta z} f(\zeta) d\zeta,$$

where integration is along a ray. The result is independent on the ray, and the resulting function F satisfies $f(\infty) = 0$ and F is analytic in $\overline{\mathbf{C}} \backslash K_f$, where K_f is a convex compact in $\mathbf{C}$ which is equal to the conjugate indicator diagram of f. The indicator diagram of f is the convex set whose support function is h_f. Moreover, K_f is the minimal (by inclusion) compact convex set K such that F has an immediate analytic continuation to $\overline{\mathbf{C}} \backslash K$.

So when $\rho = 1$, our problem has the following reformulation:

Problem 2. *Let A be the same as before, and K_a a collection of convex compact sets in the plane. Under what condition one can find analytic germs at infinity $F_1, \ldots, F_n$, with the properties $F_j(\infty) = 0$, and such that the linear combinations*

$$G_a = a_1 G_1 + \ldots + a_n G_n$$

have analytic continuations to $\overline{\mathbf{C}} \backslash K_a$ but not to larger regions of the form $\overline{\mathbf{C}} \backslash K$, with convex compacts K?

This re-formulations allows a simple solution of both problems 1 and 2 for the case when $n = 2$ and $\rho = 1$. Condition (1 is indeed necessary and sufficient in this case.

Let us denote $h = \max_A h_a$. Then condition (1) implies that $h_a \neq h$ for at most finitely many a. I know the solution of Problem 2 only for the case when $h_a \neq h$ for $n + 1$ values of a. This solution shows that (1) is not sufficient when $n \geq 3$.

Bernstein [1] generalized the Laplace transform to the case when $\rho \neq 1$. In his construction the role of $\overline{\mathbf{C}} \backslash K$ is played by some Riemann surface spread over the sphere. Unfortunately it is not clear how this generalization can lead to the solution of our problems even with $n = 2$.

Unfortunately there is no simple reformulation of completely regular growth in terms of Laplace transform.

References

[1] V. Bernstein, Sulle proprietà caratteristiche delle indicatrici di crescenza delle transcendenti intere d'ordine finito, Memorie della Reale Accademia d'Italia, Classe di Scienze Fisiche, Matematiche e Naturali, 7. Serie, 131–189 (1936).