Some definite integrals

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In this course we will make use of some important definite integrals.

1. Gamma function
$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$
Check that this integral converges for $\operatorname{Re} x > 0$. Integrating by parts, we obtain
$$\Gamma(x+1) = \int_0^\infty t^x e^{-t} dt = -\int_0^\infty t^x d(e^{-t}) = x \int_0^\infty t^{x-1} e^{-t} dt = x\Gamma(x).$$
Check that the non-integrated terms vanish. So
$$\Gamma(x+1) = x\Gamma(x). \tag{1}$$
When $x = 1$ we obtain from the definition
$$\Gamma(1) = 1. \tag{2}$$
Now for a positive integer n, using (1), (2) we obtain
$$\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2) = \ldots = (n-1)!.$$
So $\Gamma(x)$ is a continuous function on the positive ray which interpolates $(n-1)!$

For $x = 0$, the integral defining Γ diverges, and one can also see from (1) that $\Gamma(0)$ cannot be a finite number. However one can rewrite (1) as $\Gamma(x-1) = \Gamma(x)/(x-1)$, and this equation defines Γ in the half-plane $\operatorname{Re} x > -1$, except the point 0. By repeating this argument, one can define Γ for all complex values of x, except non-positive integers. At the non-positive integers $-n$, $\Gamma(-n) = \infty$ in the sense that $1/\Gamma(-n) = 0$. In fact $1/\Gamma$ is finite and continuous, actually analytic in the whole complex plane. Since $1/(\Gamma(z)\Gamma(1-z))$ is analytic in the whole complex plane and equals to 0 exactly at all integers, it is not a big surprise that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

That the constant of proportionality must be π can be seen by pluggng $z = 1/2$ and using the evaluation of $\Gamma(1/2)$ in the next section.

A very good approximation to Γ (and thus to the factorial) is given by the *Stirling formula*:

$$\Gamma(z) = e^{(z-1)\mathrm{Log} z}\sqrt{2\pi z + o(1)}, \quad |\mathrm{Arg} z| \leq \pi - \epsilon,$$

where $o(1)$ means a summand which tends to 0 as $z \to \infty$, for any fixed $\epsilon > 0$. Here

$$\mathrm{Log} z := \log|z| + i\mathrm{Arg} z, \quad -\pi < \mathrm{Arg} z < \pi,$$

the so-called Principal branch of the complex logarithm. Stirling's formula is used every time when one needs to evaluate approximately any expression involving $\Gamma(z)$ for large z.

So, in particular

$$n! = n^n e^{-n}\sqrt{2\pi n + o(1)},$$

the approximation frequently used in combinatorics and probability.

Example. Derive the asymptotic expression for the central binomial coefficient:

$$\frac{(2n)!}{(n!)^2} \sim \frac{4^n}{\sqrt{\pi n}}, \quad n \to \infty.$$

2. We will need the value $\Gamma(1/2)$.

To obtain it make the change of the variable $t = y^2$ in the integral:

$$\Gamma(1/2) = \int_0^\infty t^{-1/2} e^{-t} dt = 2\int_0^\infty e^{-y^2} dy.$$

We reduced $\Gamma(1/2)$ to another important integral. Let is give it a (temporary) name

$$I = \int_0^\infty e^{-x^2} dx.$$

To evaluate it, take the product of two such integrals, consider it as a double integral and switch to polar coordinates $r = x^2+y^2$ and θ, taking into account that $dxdy = rdr\, d\theta$:

$$\begin{aligned} I^2 &= \int_0^\infty e^{-x^2}dx \int_0^\infty e^{-y^2}dy = \int\int_{x>0,y>0} e^{-x^2-y^2}dxdy \\ &= \int_0^{\pi/2}\int_0^\infty e^{-r^2} r\, dr\, d\theta = \frac{\pi}{2}\frac{1}{2}\int_0^\infty e^{-r^2}d(r^2) = \frac{\pi}{4}. \end{aligned}$$

So we obtain these **important formulas**:

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}, \quad \text{and} \quad \Gamma(1/2) = \sqrt{\pi}. \tag{3}$$

These are two important integrals which frequently occur in mathematics and applications.

3. The following integral is called the *Fourier transform* of the function e^{-x^2}

$$F(t) = \int_{-\infty}^\infty e^{-x^2}e^{-itx}dx.$$

First of all, since our function is even,

$$\int_{-\infty}^\infty e^{-x^2}dx = \sqrt{\pi}.$$

Second, we complete the square in the exponent:

$$e^{-x^2-itx} = e^{-(x+it/2)^2-t^2/4} = e^{-(x+it/2)^2}e^{-t^2/4}.$$

Then we change the variable in our integral to $y = x + it/2$ and obtain the result

$$F(t) = \sqrt{\pi}e^{-t^2/4}.$$

To be sure, the change of the variable requires some justification, since we integrate along a line in the complex plane. This is done in the Complex Analysis courses with the help of Cauchy Theorem.

Taking real part of e^{-itx} we also obtain

$$\int_0^\infty e^{-ax^2}\cos(tx)dx = \sqrt{\frac{\pi}{2}}e^{-t^2/4}.$$

Exercise. Generalize this result:

$$\int_{-\infty}^{\infty} e^{-ax^2/2} e^{-itx} dx = \sqrt{\frac{2\pi}{a}} e^{-t^2/2a}, \quad a > 0.$$

which is entry 9 of the Table 2 on p. 223 of the book.

Application: volumes of balls and areas of spheres

Let us consider the ball $B_n(r)$ of radius r its boundary $S_n(r)$, which is the sphere of radius rs in the space $\mathbf{R}^n$.

It is clear by scaling that $(n-1)$-dimensional "area" of $S_n(r)$ is $\omega_n r^{n-1}$, where ω_n is the area of the unit sphere, and n-dimensional volume is

$$\text{volume}(B_n(r)) = \int_0^r t^{n-1} \omega_n dt = \omega_n r^n / n.$$

So one needs to find only ω.

To do this, we integrate $e^{-\|\mathbf{x}\|^2}$ over the whole space in two ways: first in Cartesian coordinates,

$$\int_{\mathbf{R}^n} e^{-\|\mathbf{x}\|^2} d\mathbf{x} = \left(\int_{-\infty}^{\infty} e^{-x^2} dx \right)^n = \pi^{n/2}$$

by (3). Second, we integrate in the spherical coordinates

$$\int_{\mathbf{R}^n} e^{-\|\mathbf{x}\|^2} d\mathbf{x} = \omega_n \int_0^{\infty} t^{n-1} e^{-t^2} dt = \frac{\omega_n}{2} \int_0^{\infty} u^{n/2-1} e^{-u} du,$$

where we made the change of the variable $u = t^2$, and the last integral is $\Gamma(n/2)$. So

$$\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)},$$

where $m = [n/2]$, the integer part of $n/2$. When $n = 2$, we obtain $\omega_2 = 2\pi$, the length of the unit circle; when $n = 3$, we use $\Gamma(3/2) = (1/2)\Gamma(1/2) = \sqrt{\pi}/2$, so $\omega_3 = 4\pi$, as Archimedes taught us.