Solution of the isochrone problem

Let the x-axis be horizontal, h-axis vertical, pointing *down*, and $x = f(h)$ be the equation of the curve. We may assume that the motion starts at the origin. Suppose that the initial speed is v_0 and the constant vertical component of velocity is V. By the Energy Concervation Law the speed $v = v(h)$ satisfies $v^2 = 2gh + v_0^2$. Denoting by α the angle between the tangent to the curve and vertical direction, we see that the vertical component of velocity is $v \cos \alpha$. On the other hand, $\tan \alpha = dx/dh$, Putting all this together, and using

$$\cos^2 \alpha = (1 + \tan^2 \alpha)^{-1},$$

we obtain the differential equation:

$$1 + (dx/dh)^2 = V^{-2}(2gh + v_0^2),$$

This equation is separable and can be reduced to evaluation of the simple integral:

$$x(h) = \int_0^h \sqrt{V^{-2}(2gh + v_0) - 1} dy = \frac{V^2}{3g} \left(\frac{2gh}{V^2} + \frac{v_0^2}{V^2} - 1 \right)^{3/2}.$$

It is clear from this formula (and also from the physical interpretation) that $v_0 \geq V$, that is the motion cannot start from rest. The curve we obtained is called a *semi-cubic parabola.*