Strings with beads and Jacobi matrices

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Let us consider a massless string of length L stretched with force σ. Suppose that the ends of the string are fixed and n beads of masses m_j are placed with equal spacing l between m_{j-1} and m_j,

$$l = \frac{L}{n+1}. \tag{1}$$

Denoting by x_j the (small) displacement of the j-th bead from the equilibrium, and by κ_j the stiffness (Hooke's constant) of the string between m_{j-1} and m_j we obtain the following system of equations:

$$m_j \frac{d^2 x_j}{dt^2} = \left(\kappa_{j+1}(x_{j+1} - x_j) - \kappa_j(x_j - x_{j-1})\right), \quad j = 1, \ldots, n.$$

Here $x_0 = x_{n+1} = 0$ because the endpoints are fixed. As every linear system describing small oscillations it is of the form

$$M\ddot{\mathbf{x}} = A\mathbf{x}.$$

where M, A are symmetric, M is positive definite and A is negative definite. In our case

$$M = \mathrm{diag}\,(m_1, \ldots, m_n),$$

and

$$A = \begin{pmatrix} -\kappa_1 - \kappa_2 & \kappa_2 & 0 & \ldots & 0 & 0 \\ \kappa_2 & -\kappa_2 - \kappa_3 & \kappa_3 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \kappa_{n-1} & -\kappa_{n-1} - \kappa_n & \kappa_n \\ 0 & 0 & 0 & 0 & \kappa_n & -\kappa_n - \kappa_{n+1} \end{pmatrix}. \tag{2}$$

Such matrices are called tri-diagonal: only the main diagonal and the two adjacent diagonals to it are different from zero. A symmetric tri-diagonal matrix is called a Jacobi matrix. Equation (2) explains a mechanical interpretation of a Jacobi matrix: it describes string with beads. There is a rich and beautiful theory of Jacobi matrices.

We consider the simplest special case where all masses and stiffness constants are equal: $m_j = m$ and $\kappa_j = \kappa$. Then $\kappa = \sigma/l$ by Hooke's Law, and the system of equations simplifies to

$$m\frac{d^2x_j}{dt^2} = \frac{\sigma}{l}\left(x_{j+1} - 2x_j + x_{j-1}\right), \quad j = 1, \ldots, n,$$

with the boundary conditions $x_0 = 0$ and $x_{n+1} = 0$. We know from general theory that eigenvalues are pure imaginary. Introducing $x_j = e^{i\omega t}u_j$, we obtain

$$u_{j+1} - \left(2 - \frac{ml}{\sigma}\omega^2\right)u_j + u_{j-1} = 0$$

with the boundary conditions $u_0 = u_{n+1} = 0$.

Let us denote for convenience

$$2 - \frac{ml}{\sigma}\omega^2 = 2b, \tag{3}$$

Then our equations are

$$u_{j+1} - 2u_j b + u_{j-1} = 0.$$

This is a difference equation with constant coefficients. To find the general solution we plug $u_j = \rho^j$ and obtain the characteristic equation

$$\rho^2 - 2b\rho + 1 = 0,$$

So $\rho = b \pm \sqrt{b^2 - 1}$. If $|b| > 1$ we have two positive distinct roots $0 < \rho_1 < \rho_2$. Then $u_j = c_1 j^{\rho_1} + c_2 j^{\rho_2}$ and it is impossible to satisfy the boundary conditions $u_0 = u_{n+1} = 0$. (Verify this!) Thus $|b| \leq 1$, in which case it is convenient to set $b = \cos\theta$, and

$$\rho_{1,2} = \cos\theta \pm i\sqrt{1 - \cos^2\theta} = \cos\theta \pm i\sin\theta = \exp(\pm i\theta)$$

So our difference equation has real general solution

$$u_j = c_1 \cos j\theta + c_2 \sin j\theta. \tag{4}$$

From the first boundary condition we obtain $c_1 = 0$ and from the second one $\sin(n+1)\theta = 0$, so

$$\theta_k = \frac{\pi k}{n+1}, \quad 1 \leq k \leq n,$$

because our solution $\sin j\theta_k$ cannot be 0 for all j (zero vector is not an eigenvector!)

From (3) we obtain the frequencies

$$\omega_k = 2\sqrt{\frac{\sigma}{ml}} \sin \frac{\pi k}{2(n+1)}, \quad 1 \leq k \leq n.$$

The eigenvectors (modes) are obtained from (4) with $c_1 = 0$ and $c_2 = 1$:

$$u_{k,j} = \sin \frac{\pi jk}{n+1}, \quad 1 \leq j \leq n.$$

We can pass to the limit when $n \to \infty$. Using (1) and $m = \rho L/n$ where ρ is the density of the string (the units are mass/length) we obtain in the limit

$$\omega_k = \sqrt{\frac{\sigma}{\rho}} \frac{\pi k}{L}, \quad k = 1, 2, 3 \ldots$$

the law discovered by Marin Mersenne and published in his book "Harmonie universelle" in 1636. The book dealt with the theory of music and musical instruments.

Notice the following consequences of this law.

1. All frequencies of a homogeneous string are integer multiples of the fundamental one,

$$\omega_1 = \sqrt{\frac{\sigma}{\rho}} \frac{\pi}{L}.$$

This fundamental frequency is called the base tone, and the rest are called overtones.

2. The base tone is inverse proportional to the length. This fact was known to the ancient Greeks, and traditionally it is credited to Pythagoras himself. Anyway, this is probably the earliest recorded result of mathematical physics, and of the exact sciences in general (6th century BC).

3. Despite a lot of research, the ancient Greeks apparently did not find the dependence of the base tone of the tension of the string (neither they

knew the dependence of the density). Probably this is because they did not have any clear concept of force, and no devices to measure it.

4. In the derivation above I followed the book by Gantmakher and Krein, Oscillation Matrices and Kernels and small vibrations of mechanical systems, Translated by AMS from the Russian original of 1941. The authors of this book credit the argument to Daniel Bernoulli (1700-1782).

5. Instead of considering a string with n beads, one can pass to the limit in the very beginning, and obtain a string of variable density, and variable elasticity. This leads to a partial differential equation instead of the ordinary differential equation. In the case of constant density and constant stiffness this equation has the simple form

$$u_{tt} = c^2 u_{xx}, \quad 0 \leq x \leq L.$$

The eigenvectors we obtained tend to the modes $u_k(x) = \sin(\pi k x/L)$. We have $c = \sqrt{\sigma/\rho}$ is the speed of the wave propagation.