Collision of a jet with a plane

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Consider a jet symmetric with respect to the x-axis, the ideal fluid moving left to right with velocity v_∞ at $-\infty$. Let $2r$ be the width of the jet. Suppose that the jet collides with a plane intersecting the x-axis at $x = 0$ and inclined to the x-axis at the angle $\alpha \in (0, \pi/2]$. We want to find the stream lines and the pressure by the jet on the plane. The outside pressure is p_0, a given constant, and we neglect all other external forces.

The region occupied by the jet will be denoted by D. The boundary ∂D consists of the upper and lower curves L^+ and L^- and the line L whose equation is $y = x \tan \alpha$. The stream ramifies at some point O on this line: one part slides up another down and back along the line L.

Let F be the complex potential of the flow. It is defined up to an additive constant; we choose this constant so that $F(O) = 0$. Then F maps conformally the region D in the z-plane onto the horizontal strip with a horizontal cut in the w-plane:

$$D' = \{w : -q' < |\Im w| < q\} \backslash \{w : w \geq 0\},$$

with some $q, q' > 0$. Let us find the constants q, q'. We have $F(z) \sim v_\infty z$ as $z \to -\infty$ in D, so

$$q + q' = 2rv_\infty. \tag{1}$$

Let a_1 and a_2 be the limit widths of the streams "up" and "down" along L, and v_1, v_2 their limit velocities at infinity. We obtain as before $q = a_1 v_1,\ q' = a_2 v_2$. Now

$$q - q' = a_1 v_1 - a_2 v_2 = 2rv_\infty \cos \alpha,$$

because the moment along the line L must be preserved: the jets slide along L without friction. From these equations we find

$$q = rv_\infty (1 + \cos \alpha), \quad q' = rv_\infty (1 - \cos \alpha).$$

Recall that $f = F'$ is the complex velocity. The pressure outside D is constant, so by Bernoulli's law the speed must be constant, so we obtain

$$|F'(z)| = v_\infty, \quad z \in L^+ \cup L^-. \tag{2}$$

On the line L we have

$$\arg F'(z) = \begin{cases} -\alpha & \text{on the upper half} \\ \pi - \alpha & \text{on the lower half.} \end{cases} \tag{3}$$

Let us consider the inverse function $G : D' \to D$, and $g = \log G'$. Then we have from (2) and (3):

$$\Re g(w) = -\log v_\infty, \quad \Im w \in \{q, -q'\}, \tag{4}$$

and

$$\begin{array}{lll} \Im g(w) = \alpha, & w = u + i0, & u > 0, \\ \Im g(w) = \alpha - \pi, & w = u - i0, & u > 0. \end{array} \tag{5}$$

This is called a *mixed boundary value problem*, the real part of the function is prescribed on one part of the boundary and the imaginary part on the complementary part.

A solution of the mixed boundary value problem with piecewise constant boundary values can be obtained by the M.V. Keldysh[1] and L.I. Sedov method as follows: just take for g the conformal map of D' onto a region bounded by the appropriate vertical and horizontal lines!

A little thinking (in which order the arcs are traced) shows that this region must be the half-strip

$$S = \{\zeta : \alpha - \pi < \Im\zeta < \alpha, \ \Re\zeta > -\log v_\infty\}.$$

This is a nice conformal map exercise. The boundary correspondence is as follows: The two "right infinities" in D' go to the 90^o corners of S, and the point $0 \in \partial D'$ goes to the infinity in ∂S.

Such map always exists by the Riemann mapping theorem, and I leave it to you to find it explicitly and to compute the pressure of the jet on the line L.

The question of uniqueness of solution, of course has not been addressed at all in this text...

[1]Besides his fine achievements in complex function theory, Keldysh also initiated the Soviet space program and was its main leader.